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editorial

Mathematicians - Keep Many Labels with You

We are born free – not as mathematicians, physicists, chemists or development monitors. After our initial studies, namely our subjects which we choose in universities, the moment we step out and join research in any subject, we don't have a single label. In any research, conclusions have to be drawn from experiments. How many measurements should be made to study wavelengths or intensities.

The minimum number of experiments is decided by statistics. One should know statistics thoroughly whether one is working in spectroscopy, nuclear physics or with nuclear reactions to study nuclear reactions. Mars spectroscopy is another. If one joins textile technology groups, one will be first initiated in statistics. One's labels may change, but not the basic concepts.

Mathematical analysis and drawing conclusions is as much important as collection of data from sophisticated experiments. Many sophisticated analysers are equipped with automatic analysers given by the manufacturer of equipments. The first-step is to cut off the computerised analysers and develop your own method. There is no substitute for self-reliance. This what is taught to us by Dr. Abdul Kalam, Dr. Bhabha, Raja Ramanna and in fact by every top scientists. The Mantra for success is Self-Reliance in every field.

Anil Ahlawat
Editor

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 153

JEE MAIN

1. A man sent 8 letters to his 8 friends. The letters were kept in addressed envelopes at random. The probability that 4 letters went to correct destinations and 4 letters went to wrong destinations is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{16}$ (d) $\frac{1}{64}$

2. The image of the curve $\arg\left(\frac{z-3}{z-i}\right)=\frac{\pi}{6}$ in the x – axis is

(a) $\arg\left(\frac{z+3}{z+i}\right)=\frac{\pi}{6}$ (b) $\arg\left(\frac{z-3}{z+i}\right)=\frac{\pi}{6}$

(c) $\arg\left(\frac{z+i}{z+3}\right)=\frac{\pi}{6}$ (d) $\arg\left(\frac{z-i}{z-3}\right)=\frac{\pi}{6}$

3. $\lim_{x\to 0}\left[\frac{2}{x^3}(\sin^{-1}x-\tan^{-1}x)\right]^{2/x^2}=e^{\lambda},\lambda=$

a) 1 (b) –1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

4. The number of positive integer solutions of $a+b+c+d\le 12$ is S. The sum of the digits of S is

(a) 17 (b) 18 (c) 19 (d) 20

5. Four equal squares are cut off from the four corners of a rectangular metal sheet of 6×9 units. The remainder is folded to form an open box. The maximum volumes of the box is

(a) $10+7\sqrt{7}$ (b) $8+9\sqrt{7}$

(c) $7+10\sqrt{7}$ (d) $12-\sqrt{7}$

JEE ADVANCED

6. If the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{k}$ and $\frac{x-1}{-k}=\frac{z-4}{2}=\frac{z-5}{1}$ are coplanar, the distance of the origin from the plane through the lines is

(a) $\frac{4}{\sqrt{6}}$ (b) $\frac{5}{\sqrt{6}}$ (c) $\sqrt{6}$ (d) $\frac{7}{\sqrt{6}}$

COMPREHENSION

Consider the curves $x^2+4y^2=4$ and $x^2+y^2=2$

7. The modulus of the slope of a common tangent is

(a) 1 (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

8. The length of a common tangent is

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) $\sqrt{3}$

INTEGER MATCH

9. Eight-digits numbers are formed with the digits 1, 2, 3, 4 such that the sum of the eight digits is 12. The sum of the digits of all such numbers is

MATCHING LIST

10. Let AD, BE, CF be the medians of triangle ABC. C_1 is the circumcircle of triangle ABC and C_2 in the circumcircle of triangle DEF.

Column-I		Column-II	
P.	C_1 and C_2 touch if $\Sigma\cos 2A$ =	1.	$-\frac{1}{2}$
Q.	C_1 and C_2 are orthogonal if $\Sigma\cos 2A$ =	2.	0
R.	C_1 passes through the centre of C_2 if $\Sigma\cos 2A$ =	3.	$\frac{1}{2}$
S.	C_1 and C_2 intersect so that common chord is greatest if $\Sigma\cos 2A$ =	4.	1
		5.	–1

	P	Q	R	S
(a)	4	5	2	3
(b)	5	4	3	2
(c)	1	2	3	4
(d)	1	3	4	2

See Solution set of Maths Musing 152 on page no. 22

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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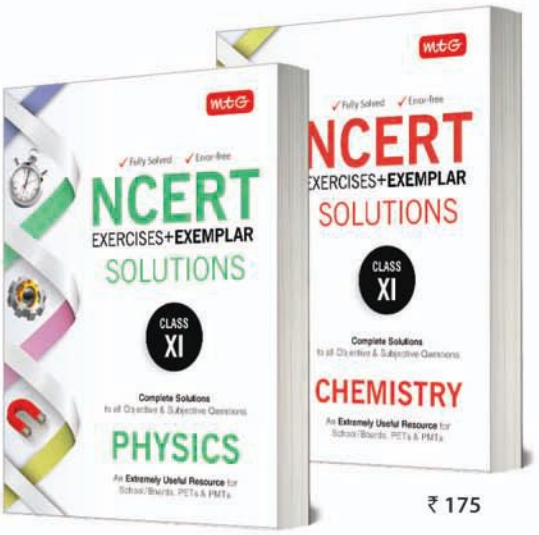

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JEE WORK OUTS

PAPER-1

SECTION-1

This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. How many solutions are there for the equation, $|\cos x - \sin x| = 2 \cos x$ in the interval $[0, 2\pi]$?

2. The number of solutions of the system $\log_2 x \log_y 2 + 1 = 0$ and $\sin x \cos y = 1 - \cos x \sin y$, which satisfy the condition $x + y < 8$ is

3. The value of $\tan 100° + \tan 125° + \tan 100° \tan 125°$ is

4. If $\cos 2\alpha = \dfrac{3\cos 2\beta - 1}{3 - \cos 2\beta}$, then $\left(\dfrac{\tan\alpha}{\tan\beta}\right)^2$ is equal to

5. If $A + B + C = 180°$, $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = k \sin\dfrac{A}{2} \sin\dfrac{B}{2} \sin\dfrac{C}{2}$, then the value of k is equal to

6. The expression $\tan 55° \tan 65° \tan 75°$ simplifies to $\cot x$, where $x \in (0, 90°)$ then x equals

7. The maximum value of $y = 2\sin^2 x - 3\sin x + 1 \; \forall \; x \in R$ is

8. Let x and y be positive real numbers and θ be an angle such that $\theta \neq \dfrac{n\pi}{2}, n \in I$. Suppose $\dfrac{\sin\theta}{x} = \dfrac{\cos\theta}{y}$ and $\dfrac{\cos^4\theta}{x^4} + \dfrac{\sin^4\theta}{y^4} = \dfrac{97\sin 2\theta}{x^3 y + y^3 x}$, the value of $\left(\dfrac{x}{y} + \dfrac{y}{x}\right)$ is equal to

SECTION-2

This section contains TEN questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

9. The vectors $\vec{a}, \vec{b}, \vec{c}$ are of the same length and the angle between any two of them is the same. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, then $\vec{c}$ is

(a) $-\dfrac{1}{3}(\hat{i} - 4\hat{j} + \hat{k})$ (b) $\hat{i} + \hat{k}$

(c) $\dfrac{1}{3}(-\hat{i} + 4\hat{j} + \hat{k})$ (d) $(\hat{i} + 2\hat{j} + 3\hat{k})$

10. The lines whose vector equations are $\vec{r} = 2\hat{i} - 3\hat{j} + 7\hat{k} + \lambda(2\hat{i} + p\hat{j} + 5\hat{k})$ and $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \mu(3\hat{i} - p\hat{j} + p\hat{k})$ are perpendicular for all values of λ and μ if

(a) $p = -6$ (b) $p = -1$

(c) $p = 1$ (d) $p = 6$

11. The vectors $\vec{a} = x\hat{i} - 2\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} + y\hat{j} - z\hat{k}$ are collinear if

(a) $x = 1, y = -2, z = -5$

(b) $x = 1/2; y = -4, z = -10$

(c) $x = -1/2, y = 4, z = 10$

(d) $x = -1; y = 2, z = 5$

12. Vector equation of the line $6x - 2 = 3y + 1 = 2z - 2$ is

(a) $\vec{r} = \hat{i} - \hat{j} + 3\hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$

(b) $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(\dfrac{1}{3}\hat{i} - \dfrac{1}{3}\hat{j} + \hat{k}\right)$

(c) $\vec{r} = \dfrac{1}{3}\hat{i} - \dfrac{1}{3}\hat{j} + \hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$

(d) $\vec{r} = -2\hat{i} + \hat{j} - 2\hat{k} + \lambda(6\hat{i} + 3\hat{j} + 2\hat{k})$

where λ, being a parameter

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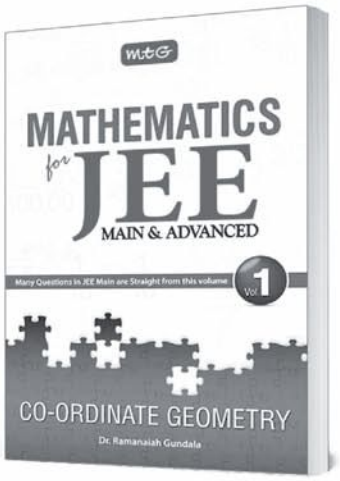

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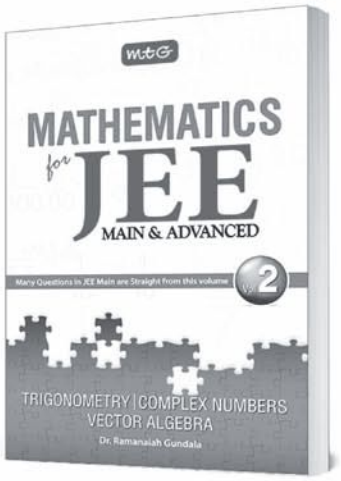

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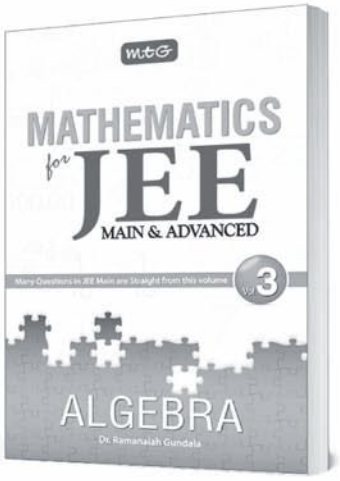

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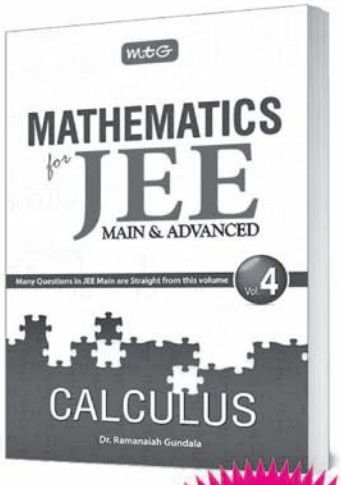

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13. The equation of the line of shortest distance between the lines $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ and $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ are

(a) $3(x-21)=3y+92=3z-32$

(b) $\frac{x-(62/3)}{1/3}=\frac{y+31}{1/3}=\frac{z-(31/3)}{1/3}$

(c) $\frac{x-21}{1/3}=\frac{y+(92/3)}{1/3}=\frac{z-(32/3)}{1/3}$

(d) $\frac{x-2}{1/3}=\frac{y+3}{1/3}=\frac{z-1}{1/3}$

14. If Δ denotes the area of the triangle ABC, then Δ is equal to

(a) $ab\sin C$ (b) $\frac{1}{2}a^2\frac{\sin B\sin C}{\sin(B+C)}$

(c) $\frac{1}{2}(a+b+c)r$ (d) $\frac{1}{2}(a+b-c)r$

15. $\sqrt{\cos 2x}+\sqrt{1+\sin 2x}=2\sqrt{\sin x+\cos x}$ if

(a) $x=n\pi-\frac{\pi}{3}$ (b) $x=2n\pi$

(c) $x=n\pi-\frac{\pi}{4}$ (d) $x=2n\pi\pm\cos^{-1}\left(-\frac{1}{5}\right)$

16. If A and B are angles such that $A+B=\frac{\pi}{3}$ and $y=\tan A\tan B$, then y can be equal to

(a) 0 (b) 2

(c) 4 (d) 5

17. The equation $\sin^4 x+\cos^4 x=a$ has a solution for

(a) all values of a (b) $a=1$

(c) $a=\frac{1}{2}$ (d) $a=\frac{3}{4}$

18. If $\theta=\tan^{-1}(2\tan^2\theta)-\tan^{-1}\left(\left(\frac{1}{3}\right)\tan\theta\right)$, then $\theta=$

(a) 0 (b) $\frac{\pi}{4}$

(c) $\tan^{-1}(-2)$ (d) none of these

SECTION-3

This section contains TWO questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D), Column II has four entries (P), (Q), (R) and (S) . Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II. The ORS contains a 4 × 4 matrix whose layout will be similar to the one shown below.

(A) P Q R S

(B) P Q R S

(C) P Q R S

(D) P Q R S

For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q), (R) and (S), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D). Marking scheme : For each entry in Column I, +2 If only the bubble(s) corresponding to all the correct match(es) is (are) darkened, 0 If none of the bubbles is darkened, –1 In all other cases.

19. Match the following :

Column I		Column II	
(A)	Maximum value of $5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)-1$ is	P.	1
(B)	$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{1}{2i^2}\right)=t$, then $\tan t=$	Q.	6
(C)	If $3\cos\theta-4\sin\theta=5$, then $3\sin\theta+4\cos\theta=$	R.	2
(D)	If $\sin A\sin B\sin C+\cos A\cos B=1$, then $2\cos(A-B)=$	S.	0

20. Match the following :

Column I		Column II	
(A)	In a ΔABC, $\cos A+\cos B+\cos C$ is equal to	P.	$\frac{a^2+b^2+c^2}{\Delta^2}$
(B)	$\frac{r_1-r}{a}+\frac{r_2-r}{b}+\frac{r_3-r}{c}=$	Q.	$1+\frac{r}{R}$
(C)	$\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=$	R.	$\frac{r_1+r_2+r_3}{s}$
(D)	$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}=$	S.	$\frac{1}{r}-\frac{1}{2R}$

PAPER-2

SECTION-1

This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. If L is the shortest distance between the lines $\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5}$ and $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}$, then determine $[L]$ ($[x]$ denotes G.I.F.).

2. Let $\vec{a}$ and $\vec{b}$ be unit vectors. If $\vec{c}$ is a vector such that $\vec{c}+(\vec{c}\times\vec{a})=\vec{b}$, then the maximum value of $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ is $A\times10^{-1}$. Find A.

3. The straight lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular then the value of $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}$ is

4. If the distance of the point $B(\hat{i}+2\hat{j}+3\hat{k})$ from the line which is passing through $A(4\hat{i}+2\hat{j}+2\hat{k})$ and which is parallel to the vector $\vec{c}=2\hat{i}+3\hat{j}+6\hat{k}$ is λ, then value of λ^2-1 is

5. Find the numbers of real solutions of the equation $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$.

6. Number of roots of the equation $\cos(\sin x)=\frac{1}{\sqrt{2}}$, $(0<x<\pi)$ is

7. Let x, y, z be real numbers such that $\cos x + \cos y + \cos z = 0$ and $\cos 3x + \cos 3y + \cos 3z = 0$ then find the maximum value of $\cos 2x \cos 2y \cos 2z$.

8. If $x + y = 2\pi/3$ and $\sin x/\sin y = 2$, then the number of values of $x \in [0, 4\pi]$ is

SECTION-2

This section contains EIGHT questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened, 0 If none of the bubbles is darkened, 2 In all other cases

9. Y-coordinate of the point of intersection of the curves, $y = \cos x$, $y = \sin 3x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ is

(a) $\cos\left(\frac{\pi}{8}\right)$ (b) $\frac{1}{\sqrt{2}}$

(c) $\sin\left(\frac{\pi}{8}\right)$ (d) $-\frac{1}{\sqrt{2}}$

10. The satisfactory value of x, where $x\in(-\pi,\pi)$ for the given equation, $(\sqrt{3}\sin x+\cos x)^{\sqrt{(\sqrt{3}\sin 2x-\cos 2x+2)}}=4$ is

(a) $\frac{\pi}{3}$ (b) $-\frac{5\pi}{3}$

(c) $-\frac{2\pi}{3}$ (d) $-\frac{\pi}{3}$

11. If $\left(\cos^2 x+\frac{1}{\cos^2 x}\right)(1+\tan^2 2y)(3+\sin 3z)=4$, then

(a) x may be a multiple of π

(b) x cannot be an odd multiple of π

(c) y can be a multiple of $\pi/2$

(d) z can be a multiple of $\pi/2$

12. The values of 'x' which satisfy the equation $2(\cos x + \cos 2x) + \sin 2x\,(1 + 2\cos x) = 2\sin x$, and $-\pi\le x\le\pi$ are

(a) $\pi, -\pi$ (b) $-(\pi/3),(\pi/3)$

(c) $-\frac{\pi}{2}$ (d) none of these

13. For $\theta\in[0,2\pi]$, the values of 'θ' and 'y' satisfying the inequality, $2^{\frac{1}{\sin^2\theta}}\cdot\sqrt{y^2-2y+2}\le2$ are

(a) $\left(\frac{\pi}{4},-1\right)$ (b) $\left(\frac{\pi}{2},1\right)$

(c) $\left(\frac{3\pi}{2},1\right)$ (d) $\left(\frac{\pi}{4},1\right)$

14. The positive integral solutions (x, y) of $\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ are

(a) (1, 2) (b) (2, 1)

(c) (2, 7) (d) (2, 2)

15. The values of x which satisfy the equation $\frac{\sqrt{3}}{2}\sin x - \cos x = \cos^2 x$, are

(a) $(2n+1)\pi$ (b) $2n\pi + \frac{\pi}{3}$

(c) $2n\pi - \frac{\pi}{3}$ (d) none of these

16. The roots of the equation $\left(\sin\frac{\pi}{6}\right)^{\cot 3x + 2\cot x}$

$= \sin^2(2\pi - x) - \cos(\pi - x) \times \sin\left(\frac{\pi}{2} + x\right)$ are

(a) $x = n\pi - \tan^{-1}\sqrt{7/5}$

(b) $x = \tan^{-1}\sqrt{7/5}$

(c) $x = \tan^{-1}\sqrt{5/7}$

(d) $x = n\pi + \tan^{-1}\sqrt{7/5}$

SECTION-3

This section contains TWO paragraphs. Based on each paragraph, there will be TWO questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is(are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

Paragraph for Q. No. 17 & 18

Let k be the length of any edge of a regular tetrahedron. (A tetrahedron whose edges are all equal in length is called a regular tetrahedron). The angle between a line and a plane is equal to the complement of the angle between the line and the normal to the plane whereas the angle between two planes is equal to the angle between the normals. Let 'D' be the origin be one of the vertex and A, B and C vertices with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively of the regular tetrahedron.

17. The angle between any edge and a face not containing the edge is

(a) $\cos^{-1}(1/2)$ (b) $\cos^{-1}(1/4)$

(c) $\cos^{-1}(1/\sqrt{3})$ (d) $\pi/3$

18. The angle between any two faces is

(a) $\cos^{-1}(1/\sqrt{3})$ (b) $\cos^{-1}(1/4)$

(c) $\pi/3$ (d) $\cos^{-1}(1/2)$

Paragraph for Q. No. 19 & 20

The circle touching BC and the two sides AB and AC produced of a triangle ABC is called the described circle opposite angle A. It's radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the described circles opposite angles B and C, respectively. They are known as the ex-radii of triangle ABC. Here Δ is the area of a triangle, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $2s = a + b + c$. Also R is the radius of the circumcircle of a triangle ABC and r is the radius of the circle inscribed in triangle ABC.

19. Suppose in a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is

(a) $\frac{1}{3}$ (b) $\frac{16}{7}$

(c) $\frac{3}{14}$ (d) $\frac{8}{19}$

20. Suppose in a triangle ABC, $8R^2 = a^2 + b^2 + c^2$, then the triangle ABC is

(a) right angled (b) isosceles

(c) equilateral (d) none of these

ANSWERS KEY

PAPER-1

1. (2) **2.** (2) **3.** (1) **4.** (2) **5.** (8)
6. (5) **7.** (6) **8.** (4) **9.** (a, b) **10.** (b, d)
11. (a, b, c , d) **12.** (c) **13.** (a, b, c)
14. (b, c) **15.** (b, c) **16.** (a, c, d) **17.** (b, c, d)
18. (a, b) **19.** (A) → Q; (B) → P; (C) → S; (D) → R
20. (A) → Q ; (B) → R ; (C) → S ; (D) → P

PAPER-2

1. (6) **2.** (5) **3.** (0) **4.** (9) **5.** (1)
6. (2) **7.** (0) **8.** (4) **9.** (a, b, c)
10. (a, c) **11.** (a, c) **12.** (a, b, d)
13. (b, c) **14.** (a, c) **15.** (a, b) **16.** (a, d) **17.** (c)
18. (d) **19.** (b) **20.** (a)

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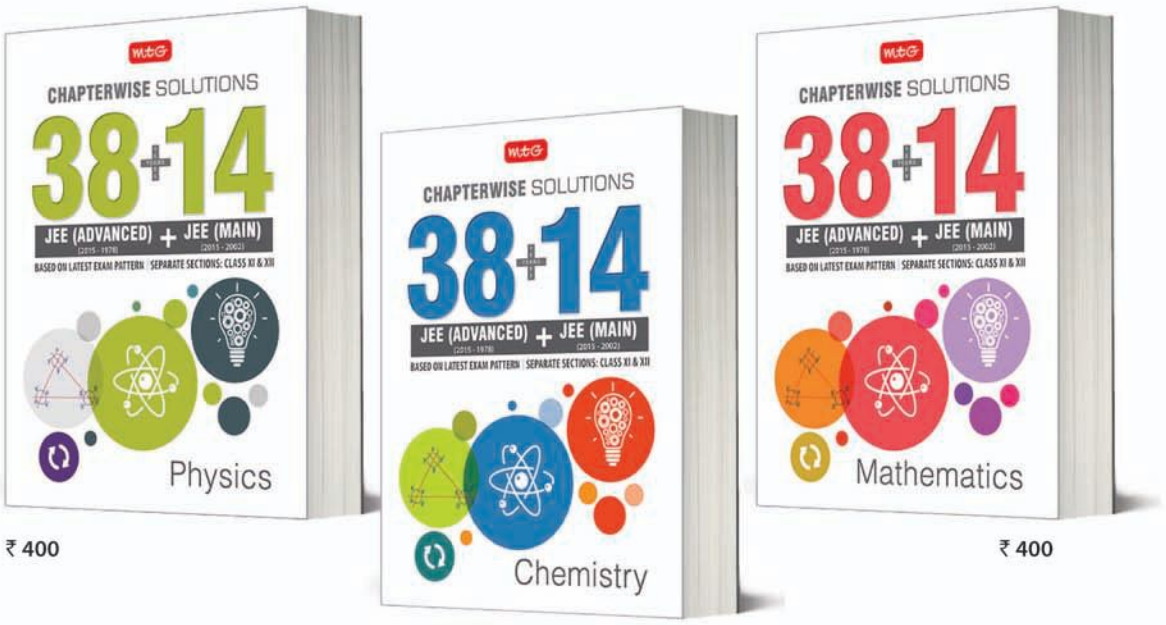

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Math Archives 10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. Discuss the monotonicity of the function g defined by $g(x) = f(x^2 - x - 10) + f(14 + x - x^2)$, $f''(x) > 0$ for all real numbers x except finite number of real numbers x, for which $f''(x) = 0$.

2. Suppose that f and g are non constant differentiable, real value functions on R. If for every $x, y \in R$, $f(x + y) = f(x) f(y) - g(x) g(y)$, $g(x + y) = g(x) f(y) + f(x) g(y)$ and $f'(0) = 0$ then prove that maximum and minimum value of the function $f^2(x) + g^2(x)$ are same for all $x \in R$.

3. Real valued function $f(x)$ satisfies the relation

$$f\left(\frac{x+y}{3}\right) = \frac{2f(x)+2f(y)-4}{6} \forall x, y \in R.$$

If $f'(0) = 2$, prove that $f(x)$ is an increasing function for all x.

4. Let $f(x) = -\frac{1}{2}(2\theta^2 - 4x - 2x^2)$, where 'θ' is a real parameter. Now let x_1, x_2 be the roots of $f(x)$ where $x_1 < x_2$. If $F(\theta) = \int_{x_1}^{x_2} f(x)dx$, find the minimum and the maximum values of $F(\theta)$ and the corresponding θ.

5. The function $f(x) = \sqrt{ax^3 + bx^2 + cx + d}$ has its non-zero local minimum and maximum values at –2 and 2 respectively. If 'a' is a root of $x^2 - x - 6 = 0$. Find possible values of a, b, c and d.

6. If $a = -1$, $b \geq 1$ and $f(x) = \frac{1}{|x|}$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval $[a, b]$, but the conclusion of the theorem is true if $b > 1 + \sqrt{2}$.

7. If $f(x) = 2x^3 - 15x^2 + 24x$, and $g(x) = \int_0^a f(x)dx + \int_0^{5-a} f(x)dx, 0 < a < 5$. Find the interval in which $g(x)$ is increasing.

8. For what value of 'a', the point of local minima of $f(x) = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ is less than 4 and point of local maxima is greater than – 2?

9. The equation $t^2 + 2xt + 4 = 0$ does not possess distinct real roots. Find the equation of the tangent of greatest slope to the curve $y = x^3 - 2x^2 + x$.

10. A point $P(x, y)$ moves on the curve $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$. For each position (x, y) of P, perpendiculars are drawn from origin upon the tangent and normal at P, the length (absolute value) of them being $p_1(x)$ and $p_2(x)$ respectively. Prove that $\frac{dp_1}{dx}.\frac{dp_2}{dx} < 0$.

SOLUTIONS

1. $f''(x) \geq 0 \Rightarrow f'(x)$ is an increasing function of x.
[$f''(x) = 0$ at finitely many values of x does not affect the increasingness of $f'(x)$]
Now $g'(x) = (2x - 1)[f'(x^2 - x - 10) - f'(14 + x - x^2)]$
Intervals of increase of g:
If $g(x)$ increases then $g'(x) \geq 0$.
$\Rightarrow 2x - 1$ and $f'(x^2 - x - 10) - f'(14 + x - x^2)$ are of same sign.
Case I : $2x - 1 \geq 0$ and $f'(x^2 - x - 10) - f'(14 + x - x^2) \geq 0$
$\Rightarrow x \geq \frac{1}{2}$ and $x^2 - x - 10 \geq 14 + x - x^2$, as f' is increasing
$\Rightarrow x \geq \frac{1}{2}$ and $x^2 - x - 12 \geq 0 \Rightarrow x \geq 4$
Case II : $2x - 1 \leq 0$ and $f'(x^2 - x - 10) - f'(14 + x - x^2) \leq 0$
$\Rightarrow x \leq \frac{1}{2}$ and $-3 \leq x \leq 4 \Rightarrow -3 \leq x \leq \frac{1}{2}$
Hence $g(x)$ increase for $x \in \left[-3, \frac{1}{2}\right] \cup [4, \infty)$
Similarly $g(x)$ decreases for $(-\infty, -3] \cup \left[\frac{1}{2}, 4\right]$

2. We have $f(x + y) = f(x) f(y) - g(x) g(y)$
Differentiating both sides w.r.t 'x' keeping 'y' constant, we get $f'(x + y) = f'(x) f(y) - g'(x) g(y)$
Putting $x = 0$ we get
$f'(y) = -g'(0) g(y)$ .....(1) (as $f'(0) = 0$)
We also have, $g(x + y) = g(x) f(y) + f(x) g(y)$

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Differentiating both sides w.r.t 'x' keeping 'y' constant, we get $g'(x) = g'(0)\, f(y)$ .....(2) (as $f'(0) = 0$)

From (1) $\times f(y)$ + (2) $\times g(y)$, we get

$f(y) f'(y) + g(y)\, g'(y) = 0$

$\Rightarrow \frac{d}{dy}\left(f^2(y) + g^2(y)\right) = 0 \Rightarrow f^2(y) + g^2(y) = \lambda$ (const)

Now putting $x = y = 0$ in both the given functional equations we get; $f(0) = f^2(0) - g^2(0)$, $g(0) = 2f(0)\, g(0)$

$\Rightarrow g(0) = 0$ or $f(0) = \frac{1}{2}$

But if $f(0) = \frac{1}{2}$, first equation gives $g^2(0) = -\frac{1}{4}$, which is not possible.

Hence $g(0) = 0$ and $f(0) = 1 \Rightarrow \lambda = 1$

Hence $f^2(x) + g^2(x) = 1, \forall\, x \in R$.

$\Rightarrow$ Maximum and minimum values of $f^2(x) + g^2(x)$ are same for all $x \in R$.

3. For $x = 0$, $y = 0$, the given equation gives

$$f(0) = \frac{4f(0) - 4}{6} \Rightarrow f(0) = -2$$

Now, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\frac{2f(3x) + 2f(3h) - 4}{6} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2f(3x) + 2f(3h) - 4 - 6f(x)}{6h}$$

For $y = 0$, the given relation yields

$$f\left(\frac{x}{3}\right) = \frac{2f(x) + 2f(0) - 4}{6}$$

$$\Rightarrow f(x) = \frac{2f(3x) - 4 - 4}{6} = \frac{f(3x) - 4}{3}$$

$$\Rightarrow f(3x) = 3f(x) + 4$$

Hence

$$f'(x) = \lim_{h\to 0} \frac{6f(x) + 8 + 2f(3h) - 4 - 6f(x)}{6h} = \lim_{h\to 0} \frac{f(3h) + 2}{3h}$$

$$= \lim_{h\to 0} \frac{f(3h) - f(0)}{3h} \Rightarrow f'(x) = 2$$

$\Rightarrow f(x) = 2x + c$, At $x = 0$, $c = -2$

$\Rightarrow f(x) = 2x - 2$

$\Rightarrow f'(x) = 2 > 0 \Rightarrow f(x)$ is always increasing.

4. Consider $g(x) = x^2 + 2x$

clearly $-\theta^2$ will be a negative number. If θ increases then $-\theta^2$ will decrease or graph of $g(x)$ will come down by the quantity $-\theta^2$. Also $F(\theta)$ is algebraic area bounded by x-axis and the curve and will be negative. So if we are increasing θ, $F(\theta)$ will decrease. Hence maximum value of $F(\theta)$ will be corresponding to $\theta = 0$ and value is equal to

$$F(\theta)_{max} = \int_{-2}^{0} (x^2 + 2x)dx = \left|\frac{x^3}{3} + x^2\right|_{-2}^{0} = \frac{-4}{3}$$

$\Rightarrow F(\theta)_{max} = -\frac{4}{3}$ for $\theta = 0$ and clearly $F(\theta)_{min}$ does not exist.

5. Since minimum occurs before maximum, so $a < 0$

Also 'a' is a root of $x^2 - x - 6 = 0 \Rightarrow a = -2$

Let $g(x) = ax^3 + bx^2 + cx + d = -2x^3 + bx^2 + cx + d$

$\Rightarrow g'(x) = -6x^2 + 2bx + c$

roots of $g'(x) = 0$ are -2 and 2

$\Rightarrow b = 0$, $c = 24$

Since minimum value is non-zero $\therefore g(-2) > 0$

$\Rightarrow d > 32$

So $a = -2$, $b = 0$, $c = 24$, $d > 32$.

6. Given, $f(x) = \frac{1}{|x|}, x \neq 0$

Let $f(0) = \lambda$, λ is definite real number.

Now $R\, f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{1}{|h|} - \lambda}{h} = \lim_{h\to 0} \frac{1 - \lambda h}{h^2} \Rightarrow \infty$$

Hence $f(x)$ is not differentiable at $x = 0$. Thus the conditions of Lagrange's Mean Value theorem are not satisfied in the interval which includes the origin. Conclusion of the Lagrange's Mean Value theorem:

$$\frac{f(b) - f(a)}{b - a} = f'(c), a < c < b$$

$$\Rightarrow \frac{\frac{1}{|b|} - \frac{1}{|a|}}{b - a} = -\frac{1}{|c|^2}$$

$$\text{or } \frac{1}{|b|} - \frac{1}{|a|} = (b - a)\left(-\frac{1}{|c|^2}\right)$$

$$\text{or } \frac{1}{|b|} - 1 = (b + 1)\left(-\frac{1}{|c|^2}\right) = -\frac{b+1}{c^2} \ (\because a = -1 \text{and} b \geq 1)$$

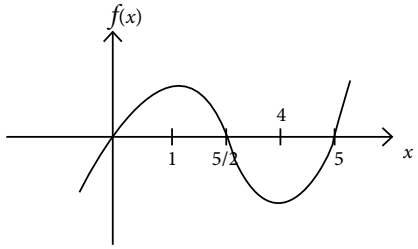
$$\text{or } c^2 = \frac{b^2 + b}{b - 1} \text{ or } \frac{b^2 + b}{b - 1} < b^2 \ (\text{as } b^2 > c^2)$$

$$\therefore \frac{b(b - 1 + \sqrt{2})(b - 1 - \sqrt{2})}{b - 1} > 0$$

$$\Rightarrow b > 1 + \sqrt{2} \ (\because b \geq 1)$$

Hence the conclusion of the L.M.V. theorem is true if $b > 1+\sqrt{2}$

7.



$f'(x) = 6x^2 - 30x + 24 = 6(x-4)(x-1)$

Graph of $f(x)$ will be as shown in fig.

$g'(a) = f(a) - f(5-a)$

if $a < 5 - a$

$\Rightarrow a < \frac{5}{2}$, then from the graph $f(a) > f(5-a)$

so $g'(0) > 0$

and if $a > \frac{5}{2}$ then $f(5-a) > f(a)$ so $g'(a) < 0$.

Hence $g(x)$ is increasing in $\left[0, \frac{5}{2}\right]$

8. $f'(x) = 3(x^2 - 2ax + a^2 - 1)$

Clearly roots of the equation $f'(x) = 0$ must be distinct and lie in the interval $(-2, 4)$

$\therefore \Delta > 0, a \in R$ ...(1)

$f'(-2) > 0 \Rightarrow a^2 + 4a + 3 > 0 \Rightarrow a < -3$ or > -1...(2)

$f'(4) > 0 \Rightarrow a^2 - 8a + 15 > 0 \Rightarrow a > 5$ or < 3 ...(3)

and $-2 < \frac{-B}{2A} < 4 \Rightarrow -2 < a < 4$ ...(4)

From (1), (2), (3) and (4), $-1 < a < 3$.

9. Since $t^2 + 2xt + 4 = 0$ does not possess distinct real roots, $4x^2 - 16 < 0 \Rightarrow -2 \le x \le 2$.

Slope of the tangent at any point (x, y) is

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

Which has max. or min.

$$\frac{d^2y}{dy^2} = 0 \Rightarrow 6x - 4 = 0 \Rightarrow x = \frac{2}{3}$$

Hence $\left(\frac{dy}{dx}\right)_{at\ x=-2} = 21, \left(\frac{dy}{dx}\right)_{x=2} = 5, \left(\frac{dy}{dx}\right)_{at\ x=2/3} = -\frac{1}{3}$

At $x = -2$, $y = -8 - 8 - 2 = -18$

$\therefore$ Equation of tangent of greatest slope is

$y - (-18) = 21(x - (-2))$

or $21x - y + 24 = 0$

10. Any point $P(x, y)$ on the curve can be represented by using parameter θ as $(a\cos^3\theta, a\sin^3\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$\therefore$ Equation of the tangent at P is

$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

or $\tan\theta\, x + y = a\sin^3\theta + a\cos^3\theta\tan\theta$

$$\Rightarrow p_1 = \left|\frac{a\sin^3\theta + a\cos^3\theta\tan\theta}{\sec\theta}\right| = \frac{1}{2}|a\sin 2\theta|$$

Also equation of the normal at P is

$y - a\sin^3\theta = \cot\theta(x - a\cos^3\theta)$

$$\Rightarrow p_2 = \left|\frac{a\cos^3\theta\cdot\cot\theta - a\sin^3\theta}{\operatorname{cosec}\theta}\right| = |a\cos 2\theta|$$

$\Rightarrow 4p_1^2 + p_2^2 = a^2 = \text{constant } \forall \text{ point } P(x, y)$

$\Rightarrow$ If p_1 increases, p_2 decreases and commonly

$\Rightarrow \frac{dp_1}{dx}$ and $\frac{dp_2}{dx}$ are of opposite signs.

$$\Rightarrow \frac{dp_1}{dx}\cdot\frac{dp_2}{dx} < 0$$

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OLYMPIAD CORNER

1. Determine the area of a triangle of sides a, b, c and semiperimeter s if

$(s-b)(s-c)=a/h,\ (s-c)(s-a)=b/k,\ (s-a)(s-b)=c/l,$
where h, k, l, are consistent given constants.

2. Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x)=\begin{cases}0 & \text{, if } 0\le x\le u\\ 1-\left(\sqrt{ux}+\sqrt{(1-u)(1-x)}\right)^2, & \text{if } u\le x\le 1\end{cases}$$

and define the sequence $\{u_n\}$ recursively as follows: $u_1 = f(1)$, and $u_n = f(u_{n-1})$ for all $n > 1$.
Show that there exists a positive integer k for which $u_k = 0$.

3. Let ABC be a triangle and let D, E, F be points on sides BC, CA, AB respectively (different from A, B, C). If $AFDE$ is inscribable in a circle show that:

$$\frac{4(DEF)}{(ABC)}\le\left(\frac{EF}{AD}\right)^2.$$

4. Let A, B, C be vectors from the circumcenter of a triangle ABC to the respective vertices. Prove that

$$\frac{(B+C)|B-C|}{|B+C|}+\frac{(C+A)|C-A|}{|C+A|}+\frac{(A+B)|A-B|}{|A+B|}=0.$$

5. For any positive integer n, evaluate $a_n/\ b_n$ where

$$a_n=\sum_{k=1}^{n}\tan^2\frac{k\pi}{2n+1},\qquad b_n=\prod_{k=1}^{n}\tan^2\frac{k\pi}{2n+1}.$$

SOLUTIONS

1. $h=\dfrac{a}{(s-b)(s-c)}=\dfrac{1}{(s-b)}+\dfrac{1}{(s-c)},$

$k=\dfrac{1}{(s-c)}+\dfrac{1}{(s-a)},$

$l=\dfrac{1}{(s-a)}+\dfrac{1}{(s-b)}.$

Hence, h, k, l, must satisfy the triangle inequality. Letting $2s' = h + k + l$, it follows by addition that

$$s'=\frac{1}{(s-a)}+\frac{1}{(s-b)}+\frac{1}{(s-c)}$$

and then

$$s-a=\frac{1}{(s'-h)},\ s-b=\frac{1}{(s'-k)},\ s-c=\frac{1}{(s'-l)}.$$

Adding the latter three equations, we get

$$s=\frac{1}{(s'-h)}+\frac{1}{(s'-k)}+\frac{1}{(s'-l)}.$$

Finally, the area of the triangle is given by

$$\Delta=\{s(s-a)(s-b)(s-c)\}^{1/2}$$
$$=\left\{\frac{\dfrac{1}{(s'-h)}+\dfrac{1}{(s'-k)}+\dfrac{1}{(s'-l)}}{(s'-h)(s'-k)(s'-l)}\right\}^{1/2}.$$

2. Note first that $u_1 = 1 - u$. Since for all $x \in [u, 1]$, $u \le x$ and $1 - x \le 1 - u$ we have

$$1-\left(\sqrt{ux}+\sqrt{(1-u)(1-x)}\right)^2$$
$$=1-ux-(1-u)(1-x)-2\sqrt{ux(1-u)(1-x)}$$
$$=u+x-2ux-2\sqrt{ux(1-u)(1-x)}$$
$$\le u+x-2ux-2u(1-x)=x-u.$$

Therefore,

$f(x)=0$ if $0\le x\le u$ ...(1)

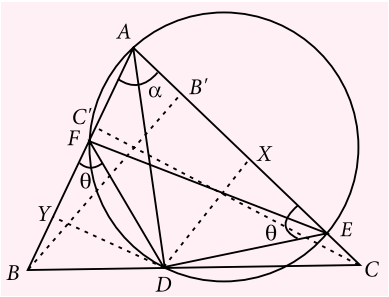
and

$f(x)\le x-u$ if $u\le x\le 1.$ ...(2)

From (2) we get $u_2 = f(u_1) - u = 1 - 2u$ if $u_1 \geq u$. An easy induction then yields $u_{n+1} = f(u_n) \leq u_n - u \leq 1 - (n+1)u$ if $u_i \geq u$ for all i =1, 2, ...,n.

Thus for sufficiently large k, we must have $u_{k-1} < u$ and then $u_k = f(u_{k-1}) = 0$ by (1).

3.



We denote the area of triangle PQR by (PQR). As A, F, D, E are concyclic, we have

$\angle EDF + \angle EAF = \pi$, so

$\sin \angle EDF = \sin \angle EAF = \sin \angle BAC$.

Therefore

$$\frac{(DEF)}{(ABC)} = \frac{\frac{1}{2} DE \cdot DF \cdot \sin \angle EDF}{\frac{1}{2} AB \cdot AC \cdot \sin \angle BAC} = \frac{DE \cdot DF}{AB \cdot AC} \qquad ...(1)$$

Let B', X be the feet of the perpendiculars from B, D to AC, respectively, and let C', Y be the feet of the perpendiculars from C, D to AB. Because BB' is parallel to DX, we get

$$\frac{DX}{BB'} = \frac{DC}{BC} \text{ and } \frac{DY}{CC'} = \frac{BD}{BC},$$

thus

$$\frac{DX}{BB'} \cdot \frac{DY}{CC'} \text{ and } \frac{DC}{BC} \cdot \frac{BD}{BC} = \frac{BD \cdot DC}{BC^2}. \qquad ...(2)$$

As $BC^2 = (BD + DC)^2$

$= (BD - DC)^2 + 4BD \cdot DC \geq 4BD \cdot DC$,

we have from (2)

$$\frac{DX \cdot DY}{BB' \cdot CC'} \leq \frac{1}{4}. \qquad ...(3)$$

We put $\angle EAF = \alpha$, and $\angle DEA = \angle DFB = \theta$, and we denote the circumradius of $AFDE$ by R, then we have $EF = 2R \sin \alpha$ and $AD = 2R \sin \theta$, so that

$$\frac{EF}{AD} = \frac{\sin \alpha}{\sin \theta} \qquad ...(4)$$

Because $BB' = AB \sin \alpha$, $CC' = AC \sin \alpha$, $DX = DE \sin\theta$, and $DY = DF \sin \theta$, we get

$$\frac{DX \cdot DY}{BB' \cdot CC'} = \frac{DE \cdot DF \sin^2 \theta}{AB \cdot AC \sin^2 \alpha}.$$

Hence we have from (3)

$$\frac{DE \cdot DF \sin^2 \theta}{AB \cdot AC \sin^2 \alpha} \leq \frac{1}{4},$$

i.e., $$\frac{4DEDF}{ABAC} \leq \frac{\sin^2 \alpha}{\sin^2 \theta} \qquad ...(5)$$

Thus we have from (1), (4) and (5)

$$\frac{4(DEF)}{(ABC)} \leq \left(\frac{EF}{AD}\right)^2.$$

4. We need to prove that

$$\frac{(B+C)|B-C|}{|B+C|} + \frac{(C+A)|C-A|}{|C+A|} + \frac{(A+B)|A-B|}{|A+B|} = 0. \qquad ...(1)$$

The lengths of

$$\frac{(B+C)}{|B+C|}|B-C|, \frac{(C+A)}{|C+A|}|C-A|, \frac{(A+B)}{|A+B|}|A-B|$$

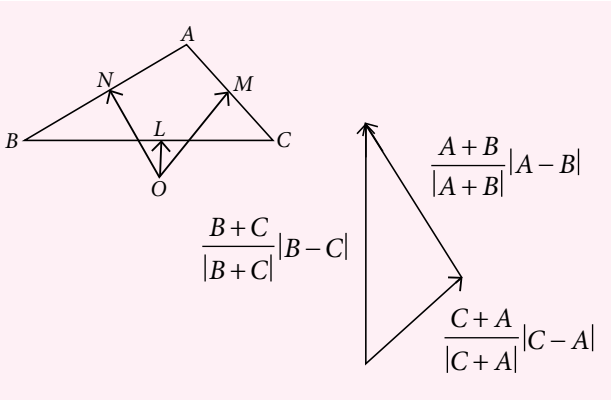
are

$|B-C| = a$, $|C-A| = b$, $|A-B| = c$,

so a triangle with sides equal to these lengths must be congruent to triangle ABC. With circumcentre as origin the directions of these vectors are perpendicular to BC, CA, AB respectively. So the given equation (1) is simply a mapping of the relation

(B – C) + (C – A) + (A – B) = 0

under rotation by 90°. This is true always provided O is internal to triangle ABC. The figure below illustrates the situation for O an external point, where evidently a sign has to be adjusted appropriately.



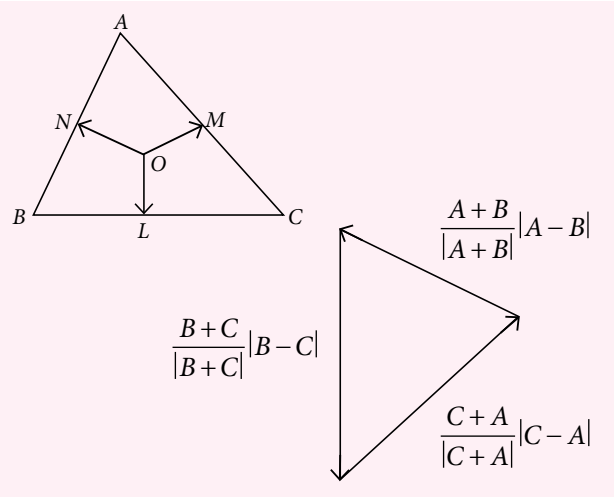
$\overrightarrow{OL}, \overrightarrow{OM}, \overrightarrow{ON}$ are in directions

$B + C$, $C + A$, $A + B$

It can be seen that for $\angle A$ obtuse one has

$$\frac{(C+A)|C-A|}{|C+A|} + \frac{(A+B)|A-B|}{|A+B|} = \frac{(B+C)|B-C|}{|B+C|}.$$

(When $\angle A = 90°$, $B + C$ vanishes and the relation degenerates and requires further interpretation.) When O is internal the situation is satisfactory, as the diagrams below show.



Now relation (1) holds.

5. Using De Moivre's theorem

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n,$$

one finds easily that

$$\sin n\theta = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}\theta \sin^{2k+1}\theta,$$

so

$$\sin(2n+1)\theta = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \cos^{2n-2k}\theta \sin^{2k+1}\theta$$

$$= \tan\theta \cos^{2n+1}\theta \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta.$$

Thus

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} \tan^{2k}\theta = 0$$

for $\theta = \frac{j\pi}{2n+1}, 1 \le j \le n,$

so $\tan^2 \frac{j\pi}{2n+1}, 1 \le j \le n,$ are the roots of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} x^k = 0,$ and thus also of

$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} x^{n-k} = 0$. Since a_n and b_n are the sum and product of the roots, respectively, we have

$$a_n = \binom{2n+1}{2} = n(2n+1) \text{ and } b_n = \binom{2n+1}{2n} = (2n+1),$$

and so $\frac{a_n}{b_n} = n.$

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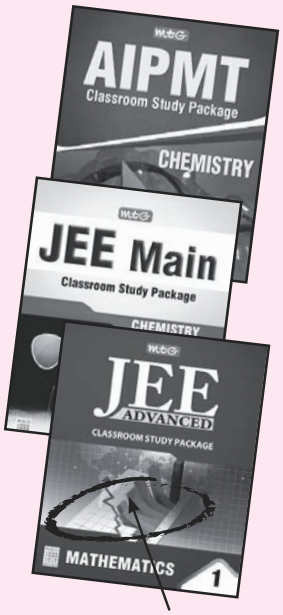

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MATHS MUSING

SOLUTION SET-152

1. **(c) :** b, g, s, n stand for boys, girls, singers, and the number of students initially.

$\frac{6n}{10} = \frac{58}{100}(n+20) \Rightarrow n = 580$

Now $s = 4 \times 58 + 20 = 252$.

2. **(d) :** $S_1S = 2ae$, $SP = a(1 - e\cos\theta)$

$S_1P = a(1 + e\cos\theta)$. $I(x, y)$ is incentre.

$x = ae\cos\theta$, $y = \frac{eb\sin\theta}{1+e} \Rightarrow \frac{x^2}{a^2} + (1+e)^2\frac{y^2}{b^2} = e^2$.

Its eccentricity $e' = \sqrt{\frac{2e}{1+e}}$

3. **(c) :** $\sin A + \sin C = 2\sin B$

$\cos\frac{B}{2} = \sin\frac{A+C}{2} \Rightarrow \sin\frac{B}{2} = \frac{1}{\sqrt{8}}$, $\cos B = \frac{3}{4}$

4. **(c) :** $A = 4\int_0^1 y\,dx = 4\int_0^1 (1 - x^{2/3})^{3/2}dx$

$= 12\int_0^{\pi/2} \cos^4\theta\sin^2\theta\,d\theta = \frac{3\pi}{8}$ if $x = \sin^3\theta$

5. **(a) :** $(1+x)^{30} = \sum_{r=0}^{30} C_r x^r$. Put $x = 1, -1, i, -i$ and add to get $C_0 + C_4 + C_8 + \ldots + C_{28} = 2^{28}$.

6. **(a, c) :** $(\vec{b} + \lambda\vec{c})\cdot\frac{\vec{a}}{|\vec{a}|} = \pm\sqrt{\frac{2}{3}} \Rightarrow \lambda = 1, -3$

$\therefore \quad 2\hat{i} + 3\hat{j} - 3\hat{k}$ and $-2\hat{i} - \hat{j} + 5\hat{k}$

7. **(c, d) :** $S = \sum_{r=1}^{5}[(4r-3)^3 - (4r-1)^3] + (21)^3$

$= -10(400 - 3) + (21)^3$

$= -3970 + 9261 = 5291$

$= 11\cdot13\cdot37$.

8. **(c) :** $S = \sum_{r=1}^{6}[(4r-3)^4 - (4r-1)^4] + (25)^4$

$= -95 \times (12)^3 + (25)^4$

$= 226465$ with digit sum 25.

9. **(3) :** $2^x = 3^y = 3\cdot2^{y+1} \Rightarrow \left(\frac{3}{2}\right)^y = 6$

$x = \log_2 3\cdot y = \log_2 3\cdot\log_{3/2}6$

$\therefore \quad a = 2, b = \frac{3}{2}$ and $ab = 3$

10. **(a) :** (P) $\frac{y}{y'} = c \Rightarrow y = k\,e^{x/c}$, exponential curve

(Q) $yy' = c \Rightarrow y^2 = 2cx + d$, parabola

(R) $\frac{y}{y'} = 2x \Rightarrow y^2 = cx$, parabola

(S) $yy' = 2x \Rightarrow y^2 = 2x^2 + c$, hyperbola.

Solution Sender of Maths Musing

SET-152

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ACE YOUR WAY CBSE XI

Series 4

COMPLEX NUMBERS AND QUADRATIC EQUATIONS & LINEAR INEQUALITIES

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

DEFINITION

A number of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ is called a complex number. It is usually denoted by z, *i.e.*, $z = x + iy$

The real and imaginary parts of a complex number z denoted by $\text{Re}(z)$ and $\text{Im}(z)$ respectively. If $z = x + iy$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$.

- A complex number z is said to be
  (i) Purely real, if $\text{Im}(z) = 0$
  (ii) Purely imaginary, if $\text{Re}(z) = 0$

Note : Order relations, "greater than" and "less than" are not defined for complex numbers.

EQUALITY OF COMPLEX NUMBERS

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $x_1 = x_2$ and $y_1 = y_2$

i.e., $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

ALGEBRA OF COMPLEX NUMBERS

1. Addition of two Complex Numbers

Let $z_1 = x + iy$ and $z_2 = u + iv$ be two complex numbers. Then the sum $z_1 + z_2 = (x + u) + i(y + v)$

Properties	
Closure law	$z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.
Commutative law	$z_1 + z_2 = z_2 + z_1$, where z_1 and z_2 are complex numbers.
Associative law	$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, where z_1, z_2 and z_3 are complex numbers.
Existence of additive identity	There exists a complex number $0 + i0$ (denoted as 0) is called additive identity.
Existence of additive inverse	For every complex number $z = x + iy$, we have $-x + i(-y)$ (denoted as $-z$) is called additive inverse or negative of z.

2. Difference of two Complex Numbers

For two complex numbers z_1 and z_2, $z_1 - z_2 = z_1 + (-z_2)$

3. Multiplication of two Complex Numbers

Let $z_1 = x + iy$ and $z_2 = u + iv$ be any two complex numbers. Then the product z_1z_2 is defined as $z_1z_2 = (xu - yv) + i(xv + yu)$

Properties	
Closure law	The product of two complex numbers *i.e.*, $z_1\ z_2$ is a complex number, where z_1 and z_2 are complex numbers.
Commutative law	For any two complex numbers z_1 and z_2, we have $z_1z_2 = z_2z_1$
Associative law	For any three complex numbers z_1, z_2, z_3, we have $(z_1z_2)z_3 = z_1(z_2z_3)$
Existence of multiplicative identity	There exists the complex number $1 + i0$ (denoted as 1), called the multiplicative identity such that $z\cdot1 = z$, for every complex number z.
Existence of multiplicative inverse	For every non-zero complex number $z = x + iy$ or $x + yi (x \neq 0, y \neq 0)$, we have the complex number $\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$ (denoted by $\frac{1}{z}$ or z^{-1}), called the multiplicative inverse of z such that $z\cdot\frac{1}{z}=1$ (the multiplicative identity)
Distributive law	For any three complex numbers z_1, z_2, z_3, we have (a) $z_1\ (z_2 + z_3) = z_1z_2 + z_1z_3$ (b) $(z_1 + z_2)\ z_3 = z_1z_3 + z_2z_3$

4. Division of two Complex Numbers

For two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by $\frac{z_1}{z_2} = z_1\frac{1}{z_2}$.

IDENTITIES OF COMPLEX NUMBERS

If z_1 and z_2 are two complex numbers, then

- $(z_1+z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$
- $(z_1-z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$
- $(z_1+z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$
- $(z_1-z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$
- $(z_1^2 - z_2^2) = (z_1+z_2)(z_1-z_2)$

MODULUS AND CONJUGATE OF A COMPLEX NUMBER

Modulus of a complex number $z = x + iy$ is denoted by $|z|$ and it is defined as $|z| = \sqrt{x^2+y^2}$

Conjugate of a complex number $z = x + iy$ is denoted by $\bar{z}$ and it is defined as $\bar{z} = x - iy$.

PROPERTIES OF MODULUS AND CONJUGATE OF A COMPLEX NUMBER

- $|z_1z_2| = |z_1|\,|z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided $|z_2| \neq 0$
- $\overline{(z_1\cdot z_2)} = (\bar{z}_1)\ (\bar{z}_2)$
- $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, (\bar{z}_2 \neq 0)$

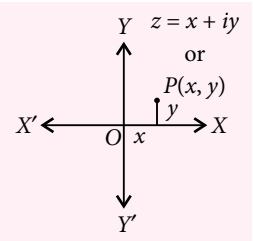
MULTIPLICATIVE INVERSE

Let $z = x + iy$ be a non-zero complex number. Then,

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} \Rightarrow z^{-1} = \frac{\bar{z}}{|z|^2}$$

ARGAND PLANE AND GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

Let O be the origin and OX and OY be the x-axis and y-axis, respectively. Then, any complex number $z = x + iy = (x, y)$ may be represented by a unique point P whose coordinates are (x, y).



The plane on which complex numbers are represented is known as the Complex plane or Argand's plane.

POLAR FORM OF A COMPLEX NUMBER

Let O be the origin and OX and OY be the x-axis and y-axis respectively. Let $z = x + iy$ be a complex number represented by the point $P(x, y)$.

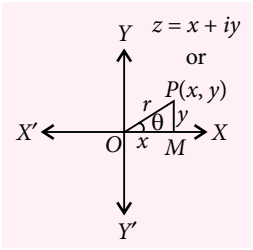
Draw $PM \perp OX$ as shown in given figure. Then, $OM = x$ and $PM = y$. Join OP. Let $OP = r$ and $\angle XOP = \theta$. Then $x = r\cos\theta$ and $y = r\sin\theta$.

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad r = \sqrt{x^2+y^2} = |z|$

Also $\quad \tan\theta = \frac{y}{x}$

This form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is called polar form of the complex number z. Angle θ is known as amplitude or agrument of z, written as arg(z).



The unique value of θ such that $-\pi < \theta \leq \pi$ for which $x = r\cos\theta$ and $y = r\sin\theta$, is known as the principal value of the argument.

The general value of the argument is $(2n\pi + \theta)$, where n is an integer and θ is the principal value of arg(z).

PROPERTIES OF ARGUMENT OF COMPLEX NUMBERS

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$ $(k = 0 \text{ or } 1 \text{ or } -1)$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$ $(k = 0 \text{ or } 1 \text{ or } -1)$

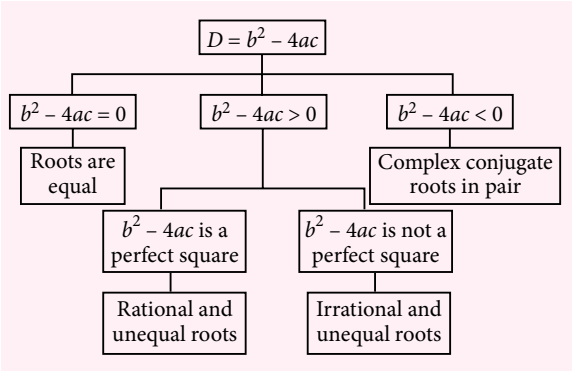
(iii) $\arg(z^n) = n\arg(z) + 2k\pi$ $(k = 0 \text{ or } 1 \text{ or } -1)$

(iv) $\arg(\bar{z}) = -\arg(z)$

QUADRATIC EQUATIONS

An equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$ is known as quadratic equation. Then the solutions of the given equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\,i}{2a}$$


D = b² – 4ac
b² – 4ac = 0
b² – 4ac > 0
b² – 4ac < 0
Roots are equal
Complex conjugate roots in pair
b² – 4ac is a perfect square
b² – 4ac is not a perfect square
Rational and unequal roots
Irrational and unequal roots

LINEAR INEQUALITIES

DEFINITION

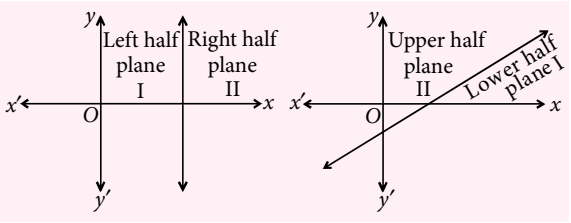
A statement involving variable(s) and the sign of inequality *i.e.*, >, <, ≥ or ≤ is called an inequation or an inequality.

	Inequalities	Definition	Examples
1.	Numerical	Inequalities which do not involve variables are called numerical inequalities.	$3 < 7$, $6 > 5$
2.	Literal	Inequalities which involve variables are called literal inequalities.	$x \geq 4$, $y > 6$, $x - y < 0$
3.	Linear	An equation of the form $ax + b < 0$, or $ax + b \geq 0$, or $ax + by > 0$ etc. are known as linear inequations.	$x + 3 < 0$, $3x + 2y > 7$
4.	Quadratic	An equation of the form $ax^2 + bx + c < 0$, or $ax^2 + bx + c \leq 0$, or $ax^2 + bx + c > 0$, or $ax^2 + bx + c \geq 0$ is known as a quadratic inequation.	$x^2 + 7x + 4 > 0$
5.	Strict	Inequalities involving the symbols '>' or '<' are called strict inequalities.	$x + y > 5$, $y < 0$
6.	Slack	Inequalities involving the symbols '≥' or '≤' are called slack inequalities.	$4x + 3y \geq 2$, $x \leq -4$

RULES FOR SOLVING INEQUALITIES

- Same number or expression may be added to or subtracted from both sides of an inequality without affecting the sign of inequality.
- Both sides of an inequality can be multiplied or divided by the same positive number without affecting the inequality sign.
- When both sides of an inequation is multiplied or divided by a negative number, then sign of inequality changes.

GRAPHICAL SOLUTION OF LINEAR INEQUALITIES IN TWO VARIABLES



The region containing all the solutions of an inequality is called the solution region.

(i) In order to identify the half plane represented by an inequality, it is just sufficient to take any point (a, b) not on line and check whether it satisfies the inequality or not.
If it satisfies, then the inequality represents the half plane and shade the region which contains the point, otherwise the inequality represents the half plane which does not contain the point within it. For the convenience, the point (0, 0) is preferred.

(ii) If an inequality is of type $ax + by \geq c$ or $ax + by \leq c$, then the point on the line $ax + by = c$ are also included in the solution region. So draw a dark line in the solution region.

(iii) If an inequality is of the type $ax + by > c$ or $ax + by < c$, then the points on the line $ax + by = c$ are not to be included in the solution region. So draw a broken or dotted line in the solution region.

Very Short Answer Type

1. Find a and b such that $2a + i\,4b$ and $2i$ represent the same complex number.

2. Express $(3 + 4i)^{-1}$ in the form $a + ib$.

3. Solve the equation $9x^2 + 16 = 0$ by factorization method.

4. Evaluate i^{-999}.

5. Solve the linear inequation $4x + 3 < 6x + 7$.

Short Answer Type

6. Write $\frac{1+7i}{(2-i)^2}$ in the polar form.

7. If $\frac{a+ib}{c+id} = x + iy$, then prove that $x^2 + y^2 = \frac{a^2+b^2}{c^2+d^2}$.

8. Solve the equation $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}\,i = 0$ using factorisation method.

9. Solve : $\frac{3x-4}{2} \geq \frac{x+1}{4} - 1$. Also, draw the graph of the solution set.

10. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form.

Long Answer Type

11. (i) Find the complex conjugate of $(2 - 3i)^2$.

(ii) If $a + ib = \frac{c+i}{c-i}$, where c is real, prove that $a^2 + b^2 = 1$ and $\frac{b}{a} = \frac{2c}{c^2-1}$.

12. Evaluate $(-2 + 2\sqrt{-3})^{1/2} + (-2 - 2\sqrt{-3})^{1/2}$

13. (i) If $z_1 = 3 + 4i$ and $z_2 = 12 - 5i$, verify $|z_1 + z_2| < |z_1| + |z_2|$

(ii) Find the conjugate and argument of $\frac{2+i}{4i+(1+i)^2}$.

14. Solve the following system of inequations graphically $3x + 2y \geq 24$, $3x + y \leq 15$, $x \geq 4$

15. (i) Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

(ii) Find the modulus and principal argument of the complex number $1 + \cos\alpha + i\sin\alpha$, $\left(0 < \alpha < \frac{\pi}{2}\right)$

SOLUTIONS

1. Given, $2a + i\,4b = 0 + i2$
Equating real and imaginary parts, we get
$2a = 0$ and $4b = 2$
$\Rightarrow a = 0$ and $b = \frac{1}{2}$

2. $(3 + 4i)^{-1} = \frac{1}{3+4i} = \frac{1}{3+4i} \times \frac{3-4i}{3-4i}$

$= \frac{3-4i}{9-16i^2} = \frac{3-4i}{9+16} = \frac{3}{25} - \frac{4}{25}i$

3. We have, $9x^2 + 16 = 0 \Rightarrow 9x^2 - 16i^2 = 0$
$\Rightarrow (3x)^2 - (4i)^2 = 0 \Rightarrow (3x + 4i)(3x - 4i) = 0$
$\Rightarrow x = -\frac{4}{3}i$, or $x = \frac{4}{3}i$

4. $i^{-999} = \frac{1}{i^{999}} = \frac{1}{(i^2)^{499}i} = \frac{1}{(-1)^{499}\cdot i} = \frac{1}{-i} = \frac{i}{-i^2} = i$

5. The given inequality is $4x + 3 < 6x + 7$.
$\Rightarrow 4x - 6x < 7 - 3$
$\Rightarrow -2x < 4 \Rightarrow x > -2$
$\Rightarrow x \in (-2, \infty)$.

6. Let $z = \frac{1+7i}{(2-i)^2}$

Now, $z = \dfrac{1+7i}{4+i^2-4i} = \dfrac{1+7i}{3-4i}$

$= \dfrac{(1+7i)(3+4i)}{(3-4i)(3+4i)} = \dfrac{-25+25i}{3^2+4^2} = -1 + i$

$\therefore \quad r = |z| = \sqrt{(-1)^2+1^2} = \sqrt{2}$

Again $z = -1 + i$

$\therefore \quad x = -1, y = 1.$

Also $\tan\theta = \left|\dfrac{y}{x}\right| = \left|\dfrac{1}{-1}\right| = 1$

$\therefore \quad \theta = \dfrac{\pi}{4}$

$\because \quad x = -1 < 0$ and $y = 1 > 0$ hence z lies in the 2^{nd} quadrant.

Hence $\arg z = \pi - \theta = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$\therefore \quad$ Polar form of z is $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$.

7. We have,

$x + iy = \dfrac{a+ib}{c+id} = \dfrac{a+ib}{c+id} \times \dfrac{c-id}{c-id}$

$= \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$= \dfrac{ac+bd}{c^2+d^2} + \dfrac{bc-ad}{c^2+d^2}i$

$\therefore \quad x = \dfrac{ac+bd}{c^2+d^2}, y = \dfrac{bc-ad}{c^2+d^2}$

[Comparing real and imaginary parts]

$\therefore \quad x^2 + y^2 = \left(\dfrac{ac+bd}{c^2+d^2}\right)^2 + \left(\dfrac{bc-ad}{c^2+d^2}\right)^2$

$= \dfrac{a^2c^2+b^2d^2+2acbd+b^2c^2+a^2d^2-2bcad}{(c^2+d^2)^2}$

$= \dfrac{(a^2c^2+b^2c^2)+(b^2d^2+a^2d^2)}{(c^2+d^2)^2}$

$= \dfrac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}$

$\therefore \quad x^2 + y^2 = \dfrac{a^2+b^2}{c^2+d^2} \qquad [\because \quad c^2 + d^2 \neq 0]$

8. We have, $x^2 - (3\sqrt{2} + 2i)x + 6\sqrt{2}\, i = 0$

$\Rightarrow (x^2 - 3\sqrt{2}\, x) - 2ix + 6\sqrt{2}\, i = 0$

$\Rightarrow x(x - 3\sqrt{2}) - 2i(x - 3\sqrt{2}) = 0$

$\Rightarrow (x - 2i)(x - 3\sqrt{2}) = 0$

$\Rightarrow x = 2i$ or $x = 3\sqrt{2}$.

Hence, the roots of the given equation are $2i$ and $3\sqrt{2}$.

9. We have, $\dfrac{3x-4}{2} \geq \dfrac{x+1}{4} - 1$

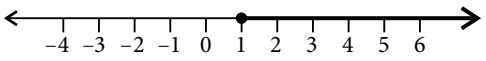
or, $\dfrac{3x-4}{2} \geq \dfrac{x-3}{4}$

or, $2(3x - 4) \geq (x - 3)$

or, $6x - 8 \geq x - 3$

or, $5x \geq 5$ or $x \geq 1$

The graphical representation of solution is given in figure.



10. $\left(\dfrac{1}{1-4i} - \dfrac{2}{1+i}\right)\left(\dfrac{3-4i}{5+i}\right)$

$= \dfrac{[1+i-2(1-4i)]}{(1-4i)(1+i)} \times \left(\dfrac{3-4i}{5+i}\right)$

$= \dfrac{(1-2)+i(1+8)}{1+i(1-4)-4(-1)} \times \dfrac{3-4i}{5+i} \qquad (\because \quad i^2 = -1)$

$= \dfrac{-1+9i}{5-3i} \times \dfrac{3-4i}{5+i}$

$= \dfrac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}$

$= \dfrac{-3+31i+36}{25-10i+3} = \dfrac{33+31i}{28-10i} \times \dfrac{28+10i}{28+10i}$

$= \dfrac{924+330i+868i+310i^2}{784-100i^2}$

$= \dfrac{924+(330+868)i-310}{784+100} \qquad (\because \quad i^2 = -1)$

$= \dfrac{614+1198i}{884} = \dfrac{307+599i}{442}$

11. (i) $(2 - 3i)^2 = (2 - 3i)(2 - 3i) = 4 - 6i - 6i + 9i^2$

$= 4 - 12i + 9(-1) = -5 - 12i$

$\therefore$ Complex conjugate of $(2 - 3i)^2$

$= \overline{(-5 - 12i)} = -5 + 12i$

(ii) We have,

$a + ib = \dfrac{c+i}{c-i} = \dfrac{c+i}{c-i} \times \dfrac{c+i}{c+i} = \dfrac{(c+i)^2}{c^2-i^2}$

$= \dfrac{c^2+2ic+i^2}{c^2+1} = \dfrac{c^2-1}{c^2+1} + i\dfrac{2c}{c^2+1}$

Equating real and imaginary parts, we get

$$a = \frac{c^2-1}{c^2+1} \text{ and } b = \frac{2c}{c^2+1}$$

$$\therefore \quad a^2 + b^2 = \left(\frac{c^2-1}{c^2+1}\right)^2 + \frac{4c^2}{(c^2+1)^2}$$

$$= \frac{c^4+1-2c^2+4c^2}{(c^2+1)^2} = \frac{(c^2+1)^2}{(c^2+1)^2} = 1$$

$$\text{Also, } \frac{b}{a} = \frac{(2c/c^2+1)}{(c^2-1)/(c^2+1)} = \frac{2c}{c^2-1}$$

12. Let $a + ib$ be a square root of $(-2 + 2\sqrt{3}\,i)$.

$\therefore \quad (a + bi)^2 = -2 + 2\sqrt{3}\,i$

$\Rightarrow \quad a^2 + 2abi + i^2b^2 = -2 + 2\sqrt{3}\,i$

$\Rightarrow \quad (a^2 - b^2) + 2abi = -2 + 2\sqrt{3}\,i$

$\therefore \quad a^2 - b^2 = -2$ …(i)

and $\quad 2ab = 2\sqrt{3}$ …(ii)

Now, $\quad (a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2$

$= (-2)^2 + (2\sqrt{3})^2 = 4 + 12 = 16$

$\therefore \quad a^2 + b^2 = \sqrt{16} = 4$ …(iii)

Solving (i) and (iii), we get

$2a^2 = 2$ or $a^2 = 1, \quad \therefore \quad a = \pm 1$

$2b^2 = 6$ or $b^2 = 3 \quad \therefore \quad b = \pm\sqrt{3}$

From (ii), $ab = \sqrt{3}$ which is positive,

$\therefore$ either $a = 1, b = \sqrt{3}$ or $a = -1, b = -\sqrt{3}$.

Hence the two square roots are $1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$.

So $(-2 + 2\sqrt{-3})^{1/2} = \pm(1 + \sqrt{3}\,i)$ …(iv)

$\therefore \quad (-2 - 2\sqrt{-3})^{1/2} = \pm(1 - \sqrt{3}\,i)$ …(v)

Adding (iv) and (v), we get

$(-2 + 2\sqrt{-3})^{1/2} + (-2 - 2\sqrt{-3})^{1/2} = \pm 2$

13. (i) We have, $z_1 = 3 + 4i$ and $z_2 = 12 - 5i$.

$z_1 + z_2 = (3 + 4i) + (12 - 5i) = 15 - i$

$$\therefore \quad |z_1 + z_2| = \sqrt{(15)^2+(-1)^2} = \sqrt{225+1} = \sqrt{226}$$

$$\text{Also, } \quad |z_1| + |z_2| = \sqrt{9+16} + \sqrt{144+25}$$

$$= 5 + 13 = 18$$

Hence $|z_1 + z_2| < |z_1| + |z_2|$

(ii) Let $z = \dfrac{2+i}{4i+(1+i)^2} = \dfrac{2+i}{4i+1+i^2+2i} = \dfrac{2+i}{6i}$

$$= \left(\frac{2+i}{6i}\right)\left(\frac{-6i}{-6i}\right) = \frac{6-12i}{36} = \frac{1}{6} - \frac{1}{3}\,i$$

$$\therefore \quad \bar{z} = \frac{1}{6} + \frac{1}{3}\,i$$

Now $\quad z = \dfrac{1}{6} - \dfrac{1}{3}\,i$

$$\therefore \quad x = \frac{1}{6},\ y = -\frac{1}{3}$$

$$\tan\theta = \left|\frac{y}{x}\right| = \left|\frac{-1/3}{1/6}\right| = |-2| = 2$$

$$\therefore \quad \theta = \tan^{-1}2,\ 0 \le \theta < \frac{\pi}{2}$$

$\because \quad x = \dfrac{1}{6} > 0$ and $y = -\dfrac{1}{3} < 0$. Hence z lies in the 4^{th} quadrant.

$\therefore$ arg $z = 2\pi - \theta = 2\pi - \tan^{-1}2$.

14. Given inequations are

$3x + 2y \ge 24$ …(i)

$3x + y \le 15$ …(ii)

and $\quad x \ge 4$ …(iii)

Corresponding equations are

$l_1 : 3x + 2y = 24$ …(iv)

$l_2 : 3x + y = 15$ …(v)

$l_3 : x = 4$ …(vi)

Line (iv) cuts x-axis at $A(8, 0)$ and y-axis at $B(0, 12)$.

Line (v) cuts x-axis at $C(5, 0)$ and y-axis at $D(0, 15)$.

Line (vi) is parallel to y-axis cutting x-axis at $E(4, 0)$.

Point $O(0, 0)$ lies on none of the lines (iv), (v) and (vi).

For $O(0, 0)$, inequation $3x + 2y \ge 24$ becomes $3\cdot0 + 2\cdot0 \ge 24$, or $0 \ge 24$, which is not true.

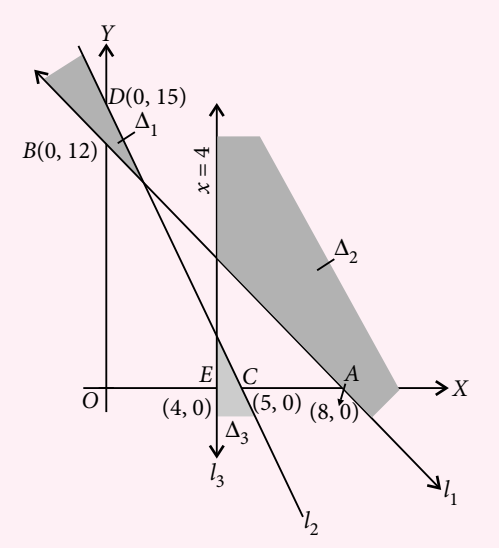
Therefore, inequation (i) represents that half-plane made by the line (iv) which does not contain the origin.

For $O(0, 0)$, inequation $3x + y \le 15$ becomes $3\cdot0 + 0 \le 15$ or $0 \le 15$, which is true.

Therefore, inequation (ii) represents that half-plane made by the line (v) which contains the origin.

Inequation $x \ge 4$ represents the half-plane on the

right side of the line $x = 4$ (which is parallel to y-axis).



Δ_1 is the region common to the half-planes representing inequations (i) and (ii).
Δ_2 is the region common to the half-planes representing inequations (i) and (iii).
Δ_3 is the region common to the half-planes representing inequations (ii) and (iii).
Clearly no point or region is common in regions $\Delta_1, \Delta_2, \Delta_3$.
Hence solution set of given system of inequations $= \phi$.

Thus no solution of the given system of inequations exists.

15. (i) Conjugate of $-6 - 24i$ is $-6 + 24i$ ...(i)

Also, $(x - iy)(3 + 5i) = 3x - 5yi^2 - 3yi + 5xi$

$= (3x + 5y) + (5x - 3y)i$ ...(ii)

From (i) and (ii), we have

$3x + 5y + (5x - 3y)i = -6 + 24i$

$\therefore \quad 3x + 5y = -6$ and $5x - 3y = 24$

or $\quad 3x + 5y + 6 = 0$ and $5x - 3y - 24 = 0$

$$\Rightarrow \quad \frac{x}{-120+18} = \frac{y}{30+72} = \frac{1}{-9-25}$$

or $$\frac{x}{-102} = \frac{y}{102} = \frac{1}{-34}$$

$$\therefore \quad x = \frac{-102}{-34} = 3, \; y = \frac{102}{-34} = -3$$

$\therefore \quad x = 3, y = -3$

(ii) $1 + \cos\alpha + i\sin\alpha = 2\cos^2\frac{\alpha}{2} + i2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

$= r(\cos\theta + i\sin\theta)$,

where $r = 2\cos\frac{\alpha}{2}$, $\theta = \frac{\alpha}{2}$.

Since $-\pi < \frac{\alpha}{2} \le \pi$, so $\frac{\alpha}{2}$ is the principal argument of given complex number and $2\cos\frac{\alpha}{2}$ is the modulus.

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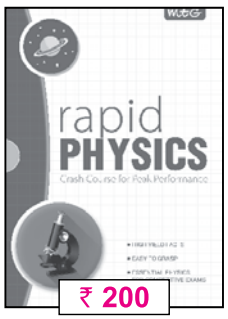

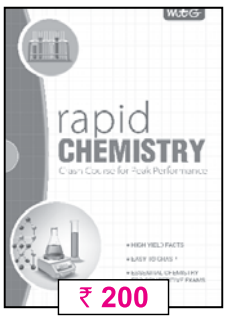

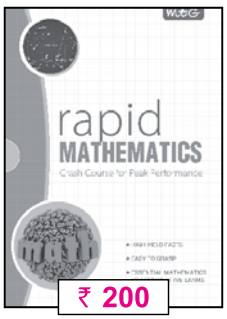

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CONCEPT BOOSTERS

Class XI

PROGRESSIONS & SERIES

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION OF A SEQUENCE

A succession of numbers $a_1, a_2, a_3, ..., a_n, ...$ formed, according to some definite rule, is called a sequence.

ARITHMETIC PROGRESSION (A.P.)

A sequence of numbers $\{a_n\}$ is called an arithmetic progression, if there is a number d, such that $d = a_n - a_{n-1}$ for all n. 'd' is called the common difference (C.D.) of the A.P.

- **Useful Formulae**

If a = first term, d = common difference and n is the number of terms, then

(a) n^{th} term is denoted by t_n and is given by

$t_n = a + (n - 1)d$.

(b) Sum of first n terms is denoted by S_n and is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

or, $$S_n = \frac{n}{2}(a + l)$$

where l = last term in the series *i.e.*,

$l = t_n = a + (n - 1)d$.

(c) If terms are given in A.P., and their sum is known, then the terms must be picked up in following way :

- For three terms in A.P., we choose them as $(a - d), a, (a + d)$
- For four terms in A.P., we choose them as $(a - 3d), (a - d), (a + d), (a + 3d)$
- For five terms in A.P., we choose them as $(a - 2d), (a - d), a, (a + d), (a + 2d)$ etc.

- **Useful Properties**
  - If $t_n = an + b$, then the series so formed is an A.P.
  - If $S_n = an^2 + bn$ then series so formed is an A.P.
  - If every term of an A.P. is increased or decreased by some quantity, the resulting terms will also be in A.P.
  - If every term of an A.P. is multiplied or divided by some non-zero quantity, the resulting terms will also be in A.P.
  - In an A.P., the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.
  - Sum and difference of corresponding terms of two A.P.'s will form an A.P.
  - If terms $a_1, a_2, ..., a_n, a_{n+1}, ..., a_{2n+1}$ are in A.P., then sum of these terms will be equal to $(2n + 1)a_{n+1}$.
  - If terms $a_1, a_2, ..., a_{2n-1}, a_{2n}$ are in A.P. The sum of these terms will be equal to $2n\left(\frac{a_n + a_{n+1}}{2}\right)$

GEOMETRIC PROGRESSION (G.P.)

A sequence of the numbers $\{a_n\}$, in which $a_1 \neq 0$, is called a geometric progression, if there is a number $r \neq 0$ such that $\frac{a_n}{a_{n-1}} = r$ for all n then r is called the common ratio (C.R.) of the G.P.

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- **Useful formulae**

If a = first term, r = common ratio and n is the number of terms, then

(a) n^{th} term, denoted by t_n, is given by

$$t_n = ar^{n-1}$$

(b) Sum of first n terms denoted by S_n is given by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ or } \frac{a(r^n-1)}{r-1}$$

In case $r = 1$, $S_n = na$.

(c) Sum of infinite terms (S_∞)

$$S_\infty = \frac{a}{1-r} \text{ (for } |r|<1 \text{ \& } r \neq 0)$$

(d) If terms are given in G.P. and their product is known, then the terms must be picked up in the following way:

For three terms in G.P., we choose them as $\frac{a}{r}, a, ar$

For four terms in G.P., we choose them as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

For five terms in G.P., we choose them as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ etc.

- **Useful properties**

(a) The product of the terms equidistant from the beginning and end is constant, and it is equal to the product of the first and the last term.

(b) If every term of a G.P. is multiplied or divided by the some non-zero quantity, the resulting progression is a G.P.

(c) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be two G.P.'s of common ratio r_1 and r_2 respectively, then $a_1b_1, a_2b_2, \ldots$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ will also form a G.P. Common ratio will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.

(d) If $a_1, a_2, a_3, \ldots$ be a G.P. of positive terms, then $\log a_1, \log a_2, \log a_3, \ldots$ will be an A.P. and conversely.

HARMONIC PROGRESSION (H.P.)

A sequence is said to be in harmonic progression, if and only if the reciprocal of its terms form an arithmetic progression.

- If a, b are first two terms of an H.P., then

$$t_n = \frac{1}{\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right)}$$

- There is no formula for sum of n terms of an H.P.
- If terms are given in H.P. then the terms could be picked up in the following way :

For three terms in H.P, we choose them as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

For four terms in H.P, we choose them as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$

For five terms in H.P, we choose them as $\frac{1}{a-2d}, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$

- **Useful properties**

If every term of a H.P. is multiplied or divided by some non zero fixed quantity, the resulting progression is a H.P.

INSERTION OF MEANS BETWEEN TWO NUMBERS

Let a and b be two given numbers.

- **Arithmetic means**

If three terms are in A.P. then the middle term is called the arithmetic mean (A.M.) between the other two *i.e.* if a, b, c are in A.P. then $b = \frac{a+c}{2}$ is the A.M. of a and c.

If $a, A_1, A_2, \ldots, A_n, b$ are in A.P., then $A_1, A_2, \ldots, A_n$ are called n A.M.'s between a and b. If d is the common difference, then

$$b = a + (n+2-1)d \Rightarrow d = \frac{b-a}{n+1}$$

$$A_i = a + id = a + i\frac{b-a}{n+1} = \frac{a(n+1-i)+ib}{n+1},$$

$i = 1, 2, 3, \ldots, n$

Note: The sum of n-A.M's,

i.e., $A_1 + A_2 + \ldots + A_n = \frac{n}{2}(a+b)$

- **Geometric means**

If three terms are in G.P. then the middle term is called the geometric mean (G.M.) between the two. So if a, b, c are in G.P. then $b = \sqrt{ac}$ or $b = -\sqrt{ac}$ corresponding to a & b both are positive or negative respectively.

If $a, G_1, G_2, \ldots, G_n, b$ are in G.P., then $G_1, G_2, \ldots, G_n$ are called n G.M.s between a and b. If r is the common ratio, then

$$b = ar^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$$

$$G_i = ar^i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}} = a^{\frac{n+1-i}{n+1}} \cdot b^{\frac{i}{n+1}},\ i = 1, 2, ..., n$$

Note: The product of n-G.M.'s *i.e.*,

$G_1 G_2 ... G_n = (\sqrt{ab})^n$

- **Harmonic mean**

If a & b are two non-zero numbers, then the harmonic mean of a & b is a number H such that

a, H, b are in H.P. & $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$ or $H = \frac{2ab}{a+b}$

If $a, H_1, H_2, ..., H_n, b$ are in H.P., then $H_1, H_2, ..., H_n$ are called n H.M.'s between a and b. If d is the common difference of the corresponding A.P., then

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)d \implies d = \frac{a-b}{ab(n+1)}$$

$$\frac{1}{H_i} = \frac{1}{a} + id = \frac{1}{a} + i\frac{a-b}{ab(n+1)},$$

$$H_i = \frac{ab(n+1)}{b(n-i+1)+ia},\ i = 1, 2, 3, ..., n$$

- Term t_{n+1} is the arithmetic, geometric or harmonic mean of t_1 & t_{2n+1} according as the terms t_1, t_{n+1} t_{2n+1} are in A.P., G.P. or H.P. respectively.

ARITHMETICO-GEOMETRIC SERIES

A series whose each term is formed, by multiplying corresponding terms of an A.P. and a G.P., is called an Arithmetico-geometric series.

e.g. $1 + 2x + 3x^2 + 4x^3 + ...$;

$a + (a + d)r + (a + 2d)r^2 + ...$

- **Summation of n terms of an Arithmetico-Geometric Series**

Let $S_n = a + (a + d)r + (a + 2d)r^2 + ...$

$+ [a + (n - 1)d]r^{n-1}$, $d \neq 0$, $r \neq 0, 1$

Multiply by 'r' and rewrite the series in the following way

$rS_n = ar + (a + d)r^2 + (a + 2d)r^3 + ...$

$+ [a + (n - 2)d]r^{n-1} + [a + (n - 1)d]r^n$

on subtraction,

$S_n(1 - r) = a + d(r + r^2 + ... + r^{n-1})$

$- [a + (n - 1)d]r^n$

or, $S_n(1-r) = a + \frac{dr(1-r^{n-1})}{1-r} - [a+(n-1)d].r^n$

or, $S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]}{1-r}.r^n$

- **Summation of infinite series**

If $|r| < 1$, then r^n, $r^{n-1} \to 0$, as $n \to \infty$.

Thus $S_\infty = S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

SUM OF MISCELLANEOUS SERIES

- **Difference Method**

Suppose a_1, a_2, a_3, ... is a sequence such that the sequence $a_2 - a_1$, $a_3 - a_2$, ... is either an A.P. or G.P. The n^{th} term 'a_n' of this sequence is obtained as follows :

$S = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$

$S = a_1 + a_2 + ... + a_{n-2} + a_{n-1} + a_n$

$\implies a_n = a_1 + [(a_2 - a_1) + (a_3 - a_2) + ...$

$+ (a_n - a_{n-1})]$

Since the terms within the brackets are either in an A.P. or in a G.P., we can find the value of a_n, the n^{th} term. We can now find the sum of the n terms of the sequence as $S = \sum_{k=1}^{n} a_k$

- **$V_n - V_{n-1}$ Method**

Let T_1, T_2, T_3, ... be the terms of a sequence. If there exists a sequence V_1, V_2, V_3, ... satisfying $T_k = V_k - V_{k-1}$, $k \geq 1$,

then $S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (V_k - V_{k-1}) = V_n - V_0$.

INEQUALITIES

- **A.M. $\geq$ G.M. $\geq$ H.M.**

Let a_1, a_2, ..., a_n be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as

$A = \frac{a_1 + a_2 + ... + a_n}{n}$, $G = (a_1 a_2 .... a_n)^{1/n}$ and

$$H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if

$a_1 = a_2 = ... = a_n$

- **Weighted means**

Let a_1, a_2, ..., a_n be n positive real numbers and w_1, w_2, ..., w_n be n positive rational numbers. Then we define weighted Arithmetic mean (A^*), weighted Geometric mean (G^*) and weighted harmonic mean (H^*) as

$$A^* = \frac{a_1 w_1 + a_2 w_2 + ... + a_n w_n}{w_1 + w_2 + ... + w_n},$$

$$G^* = (a_1^{w_1}.a_2^{w_2}...a_n^{w_n})^{\frac{1}{w_1+w_2+...+w_n}}$$

and $H^* = \dfrac{w_1+w_2+...+w_n}{\frac{w_1}{a_1}+\frac{w_2}{a_2}+...+\frac{w_n}{a_n}}$

$A^* \geq G^* \geq H^*$. Moreover equality holds at either place if & only if $a_1 = a_2 = ... = a_n$

- **Cauchy's Schwarz Inequality:**

If $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ are $2n$ real numbers, then

$(a_1b_1 + a_2b_2 + ... + a_n b_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$

with the equality holding if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$$

ARITHMETIC MEAN OF m^{th} POWER

Let $a_1, a_2, ... , a_n$ be n positive real numbers and let m be a real number, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} \geq \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m,$$

if $m \in R - [0, 1]$

However if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} \leq \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m$$

Obviously if $m \in \{0, 1\}$, then

$$\frac{a_1^m + a_2^m + ... + a_n^m}{n} = \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m$$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. If $a_n = \sum_{k=0}^{n} \frac{(\log_e 10)^n}{k!(n-k)!}$ for $n \geq 0$ then $a_0 + a_1 + a_2 + a_3 + ...$ upto ∞ is equal to

(a) 10 (b) 10^2 (c) 10^3 (d) 10^4

2. If a_1 is the greatest value of $f(x)$, where $f(x) = \left(\frac{1}{2+[\sin x]}\right)$ (where [.] denotes greatest integer function) and $a_{n+1} = \frac{(-1)^{n+2}}{(n+1)} + a_n$, then $\lim_{n\to\infty} (a_n)$ is

(a) 1 (b) e^2

(c) ln2 (d) ln3

3. The sequence $\{x_k\}$ is defined by $x_{k+1} = x_k^2 + x_k$ and $x_1 = 1/2$. Then $\left[\frac{1}{x_1+1} + \frac{1}{x_2+1} + ... + \frac{1}{x_{100}+1}\right]$ (where [.] denotes the greatest integer function) is equal to

(a) 0 (b) 2 (c) 4 (d) 1

4. If $S_n = \sum_{r=1}^{n} t_r = \frac{1}{6}n(2n^2+9n+13)$ then $\sum_{r=1}^{n} \sqrt{t_r}$ equals to

(a) $\frac{1}{2}n(n+1)$ (b) $\frac{1}{2}n(n+2)$

(c) $\frac{1}{2}n(n+3)$ (d) $\frac{1}{2}n(n+5)$

5. $\{a_n\}$ and $\{b_n\}$ be two sequences given by $a_n = (x)^{1/2^n} + (y)^{1/2^n}$ and $b_n = (x)^{1/2^n} - (y)^{1/2^n}$ for all $n \in N$, then $a_1\, a_2\, a_3 \ldots a_n$ is equal to

(a) $x - y$ (b) $\frac{x+y}{b_n}$

(c) $\frac{x-y}{b_n}$ (d) $\frac{xy}{b_n}$

6. The sum of the series $\frac{9}{5^2 \cdot 2 \cdot 1} + \frac{13}{5^3 \cdot 3 \cdot 2} + \frac{17}{5^4 \cdot 4 \cdot 3} + ...$ upto infinity is

(a) 1 (b) 9/5 (c) 1/5 (d) 2/5

7. Sum of first n terms of a sequence is given by $3S_n = T_n^2 + 3T_n - 2$, $(T_n > 0)$, where T_n is the n^{th} term of sequence, then the value of T_2^2 is

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$

(c) $2+3\sqrt{2}$ (d) $3+2\sqrt{2}$

8. If a, b, c are three distinct numbers such that a, b, c are in A.P. and $b - a, c - b, a$ are in G.P., then $a : b : c$ are in

(a) 2 : 3 : 4 (b) 3 : 4 : 5

(c) 1 : 3 : 5 (d) 1 : 2 : 3

9. The sum to n terms of the series $\frac{1}{\lfloor 1}\left(\frac{1}{2}\right) + \frac{1\cdot 3}{\lfloor 2}\left(\frac{1}{2}\right)^2 + \frac{1\cdot 3\cdot 5}{\lfloor 3}\left(\frac{1}{2}\right)^3 + ...$ upto n terms is

(a) $\frac{1\cdot 3\cdot 5 .... (2n-1)(2n+1)}{2^n \lfloor n} - 1$

(b) $1 - \frac{1\cdot 3\cdot 5 ..... (2n-1)}{\lfloor n \ \lfloor n}$

(c) $1 - \frac{1\cdot 3\cdot 5 .....(2n-3)}{2^{n-1}\lfloor n-1}$

(d) $\frac{1\cdot 3\cdot 5 ..... (2n-3)}{2^{n-1}\lfloor n-1}$

10. The ratio of sum of first three terms of a G.P. to the sum of first six terms is 64 : 91, the common ratio of G.P. is
(a) 1/4 (b) 3/4 (c) 5/4 (d) 7/4

11. Sum of the series $3 + 5 + 9 + 17 + 33 + ...$ to n terms is
(a) $2^{n+1} - n - 2$ (b) $2^{n+1} + n - 2$
(c) $2^n + n - 2$ (d) $2^{n+1} - n + 2$

12. If one A.M. A and two G.M.s, p and q be inserted between two numbers a and b, then which of the following holds good ?
(a) $a^3 + b^3 = 2Apq$ (b) $p^3 + q^3 = 2Apq$
(c) $a^3 + b^3 = 2Aab$ (d) none of these

13. If fourth term of a G.P. is 3, the product of the first seven terms is
(a) 3^4 (b) 3^7 (c) 7^4 (d) 4^7

14. The number of divisors of 6912, 52488, 32000 are in
(a) A.P. only (b) G.P. only
(c) A.P. , G.P. & H.P. (d) none of these

15. If $1, \log_{81}(3^x + 48), \log_9\left(3^x - \frac{8}{3}\right)$ are in A.P., then the value of x equals
(a) 9 (b) 6 (c) 2 (d) 4

16. If a, b, c are in H.P., then $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in
(a) A.P. (b) H.P.
(c) G.P. (d) none of these

17. If the arithmetic mean of two positive numbers a and b $(a > b)$ is twice their G.M., then $a : b$ is
(a) $6+\sqrt{7} : 6-\sqrt{7}$ (b) $2+\sqrt{3} : 2-\sqrt{3}$
(c) $5+\sqrt{6} : 5-\sqrt{6}$ (d) none of these

18. The number of terms common between the two series $2 + 5 + 8 + \ldots$upto 50 terms and the series $3 + 5 + 7 + 9 + \ldots$.upto 60 terms is
(a) 24 (b) 26
(c) 25 (d) none of these

19. If a, b, c are three positive numbers, then the minimum value of the expression $\frac{ab(a+b)+bc(b+c)+ca(c+a)}{bca}$ is
(a) 3 (b) 4 (c) 6 (d) 1

20. Sum of n terms of the series
$ab + (a + 1)(b + 1) + (a + 2)(b + 2) + ... + (a + (n - 1))(b + (n - 1))$
if $ab = \frac{1}{6}$ and $(a+b) = \frac{1}{3}$, is
(a) $\frac{n}{6}(1-2n)^2$ (b) $\frac{n}{6}(1+n-2n^2)$
(c) $\frac{n}{6}(1-2n+2n^2)$ (d) none of these

21. If $\log(a + c), \log(a + b), \log(b + c)$ are in A.P. and a, c, b are in H.P, then the value of $a + b$ is (given $a, b, c > 0$)
(a) $2c$ (b) $3c$ (c) $4c$ (d) $6c$

22. If a, b, c, are in A.P. and p, p' are respectively A.M. and G.M. between a and b while q, q' are respectively AM. and G.M. between b and c, then
(a) $p^2 + q^2 = p'^2 + q'^2$ (b) $pq = p'q'$
(c) $p^2 - q^2 = p'^2 - q'^2$ (d) $p^2 + p'^2 = q^2 + q'^2$

23. a, b, c are positive integers forming an increasing G.P. whose common ratio is a natural number, $b - a$ is cube of a natural number and $\log_6 a + \log_6 b + \log_6 c = 6$, then $a + b + c =$
(a) 100 (b) 111 (c) 122 (d) 189

24. If S, P and R are the sum, product and sum of the reciprocals of n terms of an increasing G.P. and $S_n = R^n \cdot P^k$, then k is equal to
(a) 1 (b) 2
(c) 3 (d) none of these

25. If 11 A.M.'s are inserted between 28 and 10, then number of integral A.M's is
(a) 5 (b) 6 (c) 7 (d) 8

26. If a, b, c are in H.P., then $\frac{1}{b-a} + \frac{1}{b-c}$ is equal to
(a) $\frac{2}{b}$ (b) $\frac{2}{a+c}$
(c) $\frac{1}{a+c}$ (d) none of these

27. Let a_n be the n^{th} term of an A.P. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the A.P. is
(a) $\alpha - \beta$ (b) $\beta - \alpha$
(c) $\frac{\alpha-\beta}{2}$ (d) none of these

28. If a, b, c, d are in G.P., then $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$ equals to
(a) $ab + bc + cd$ (b) $(ab + bc + cd)^2$
(c) $(ab + bc + cd)^4$ (d) none of these

29. If a_1, a_2, a_3, a_4, a_5 are in H.P., then $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5$ is equal to
(a) $2a_1a_5$ (b) $3a_1a_5$
(c) $4a_1a_5$ (d) -4

SECTION-II

Multiple Correct Answer Type

30. If a, b, c, d are four unequal positive numbers which are in A.P. then

(a) $\frac{1}{a}+\frac{1}{d}=\frac{1}{b}+\frac{1}{c}$ (b) $\frac{1}{a}+\frac{1}{d}<\frac{1}{b}+\frac{1}{c}$

(c) $\frac{1}{a}+\frac{1}{d}>\frac{1}{b}+\frac{1}{c}$ (d) $\frac{1}{b}+\frac{1}{c}>\frac{4}{a+d}$

31. If the first and $(2n-1)^{th}$ terms of an A.P., G.P. and H.P. of positive terms are equal and their n^{th} terms are a, b, c respectively, then

(a) $a = b = c$ (b) $a \geq b \geq c$

(c) $b^2 = ac$ (d) $a + c = 2b$

32. In a G.P., the product of first four terms is 4 and the second term is the reciprocal of the fourth term. The sum of infinite terms of the G.P. can be

(a) -8 (b) $-8/3$ (c) $8/3$ (d) 8

33. If $a_1, a_2, a_3, ..., a_n$ is a sequence of positive numbers which are in A.P. with common difference 'd' and $a_1 + a_4 + a_7 + ... + a_{16} = 147$ then

(a) $a_1 + a_6 + a_{11} + a_{16} = 98$

(b) $a_1 + a_{16} = 49$

(c) $a_1 + a_4 + a_7 + ... + a_{16} = 6a_1 + 45d$

(d) Maximum value of $a_1a_2 .... a_{16}$ is $\left(\frac{49}{2}\right)^{16}$.

34. If $T_r=\frac{1}{r\sqrt{r+1}+(r+1)\sqrt{r}}$, then (here $r \in N$)

(a) $T_r > T_{r+1}$ (b) $T_r < T_{r+1}$

(c) $\sum_{r=1}^{99} T_r=\frac{9}{10}$ (d) $\sum_{r=1}^{n} T_r<1$

35. If $S_{(n)}=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}, (n\in N)$, then $S_{(1)} + S_{(2)} + ... + S_{(n-1)}$ is equal to

(a) $nS_{(n)} - n$ (b) $nS_{(n)} - 1$

(c) $(n-1)S_{(n-1)} - n$ (d) $nS_{(n-1)} - n + 1$

36. If $a_i > 0$ for $i = 1, 2, \ldots, n$ and $a_1a_2 .... a_n = 1$, then $(2 + a_1)(2 + a_2) .... (2 + a_n)$ is greater than

(a) $2^{n/2}$ (b) $2^{3n/2}$

(c) 2^{2n} (d) none of these

37. The p^{th} term T_p of H.P. is $q(p + q)$ and q^{th} term T_q is $p(p + q)$ when $p > 1, q > 1$ $(p \neq q)$ then

(a) $T_{p+q} = pq$ (b) $T_{pq} = p + q$

(c) $T_{p+q} > T_{pq}$ (d) $T_{pq} > T_{p+q}$

38. If b_1, b_2, b_3 are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(a) $r > 3$ (b) $1<r<\frac{3}{2}$

(c) $r < 1$ (d) none of these

39. If a, b, c are in H.P., then

(a) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

(b) $\frac{2}{b}=\frac{1}{b-a}+\frac{1}{b-c}$

(c) $a-\frac{b}{2}, \frac{b}{2}, c-\frac{b}{2}$ are in G.P.

(d) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

SECTION-III

Comprehension Type

Paragraph for Question No. 40 to 42

If $x_1, x_2, x_3, ..., x_n$ are n positive real numbers, then A.M. $\geq$ G.M. *i.e.*, $\frac{x_1+x_2+...+x_n}{n} \geq \sqrt[n]{x_1x_2....x_n}$

40. The minimum value of the function $4^{\sin^{-1}x} + 4^{\cos^{-1}x}$ $(-1 \leq x \leq 1)$ and the value of x, where it is attained is

(a) $2\cdot 2^{\pi/2}$ at $x=\frac{1}{2}$ (b) $2\cdot 2^{\pi/4}$ at $x=\frac{1}{\sqrt{2}}$

(c) $2\cdot 2^{\pi/2}$ at $x=\frac{1}{\sqrt{2}}$ (d) $1+4^{\pi/2}$ at $x=0$

41. Which of the following inequalities is not true?

(a) $\frac{x^2+3}{\sqrt{x^2+2}} \geq 2 \quad (x \in R)$

(b) $x^2 + y^2 + 1 \geq xy + x + y$ $(x, y \in R)$

(c) $\frac{x^3+x+2}{x} \geq 4 \, (x > 0)$

(d) $x^2+\frac{1}{x^2}+4 \leq 3\left|x+\frac{1}{x}\right| \, (x \neq 0)$

42. If the equation $x^4 - 4x^3 + ax^2 - bx + 1 = 0$ has four positive roots, then $a + b$ is equal to

(a) 0 (b) 4 (c) 6 (d) 10

Paragraph for Question No. 43 to 45

If a sequence or series is not a direct form of an A.P., G.P., etc. then its n^{th} term cannot be determined. In such cases, we use the following steps to find the n^{th} term of the given sequence.

Step-I : Find the differences between the successive terms of the given sequence. If these differences are in A.P., then take $T_n = an^2 + bn + c$, where a, b, c are constants.

Step-II : If the successive differences found in step I are in G.P. with common ratio r, then take $T_n = a + bn + cr^{n-1}$, where a, b, c are constants.

Step-III : If the second successive differences (Differences of the differences) in step I are in A.P., then take $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constants.

Step-IV : If the second successive differences (Differences of the differences) in step I are in G.P., then take $T_n = an^2 + bn + c + dr^{n-1}$, where a, b, c, d are constants.

Now let sequences :

A : 1, 6, 18, 40, 75, 126, ...

B : 1, 1, 6, 26, 91, 291, ...

C : ln2, ln4, ln32, ln1024 .....

43. If the n^{th} term of the sequence A is $T_n = an^3 + bn^2 + cn + d$ then the value of $6a + 2b - d$ is

(a) ln 2 (b) 2 (c) ln 8 (d) 4

44. For the sequence 1, 1, 6, 26, 91, 291, ……., find S_{50} where $S_{50} = \sum_{r=1}^{50} T_r$

(a) $\frac{5}{8}(3^{50}-1)-3075$ (b) $\frac{5}{8}(3^{50}-1)-5075$

(c) $\frac{5}{8}(3^{50}-1)-1275$ (d) none of these

45. The sum of the series

$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + ... + n\cdot 1$ is

(a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n+1)(n+2)}{3}$

(c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{n(n+1)(2n+1)}{3}$

Paragraph for Question No. 46 to 48

In a sequence of $(4n + 1)$ terms, the 1^{st} $(2n + 1)$ terms are in A.P. whose common difference is 2 and the last $(2n + 1)$ terms are in G.P. whose common ratio is 1/2. If the middle terms of the A.P. and G.P. are equal, then

46. Middle term of the sequence is

(a) $\frac{n\cdot 2^{n+1}}{2^n-1}$ (b) $\frac{n\cdot 2^{n+1}}{2^{2n}-1}$

(c) $n2^n$ (d) $(n+1)2^{n+1}$

47. First term of the sequence is

(a) $\frac{4n+2n\cdot 2^n}{2^n-1}$ (b) $\frac{4n-2n\cdot 2^n}{2^n-1}$

(c) $\frac{2n-n\cdot 2^n}{2^n-1}$ (d) $\frac{2n+n\cdot 2^n}{2^n-1}$

48. Middle term of the G.P. is

(a) $\frac{2^n}{2^n-1}$ (b) $\frac{n2^n}{2^n-1}$ (c) $\frac{n}{2^n-1}$ (d) $\frac{2n}{2^n-1}$

SECTION-IV

Matrix-Match Type

49. Match the following :

Column-I		Column-II	
(A)	If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P. then $p-q, q-r, r-s$	(p)	are all equal
(B)	If $\ln x, \ln y, \ln z$ $(x, y, z > 1)$ are in G.P then $2x + \ln(\ln x)$, $3x + \ln(\ln y)$, $4x + \ln(\ln z)$	(q)	are in A.P.
(C)	If $n!$, $3 \times n!$ and $(n+1)!$ are in G.P. then $n!$, $5 \times n!$ and $(n+1)!$	(r)	are in G.P.
(D)	If the arithmetic mean of $(b-c)^2$, $(c-a)^2$ and $(a-b)^2$ is same as that of $(b+c-2a)^2$, $(c+a-2b)^2$, $(a+b-2c)^2$ then a, b, c	(s)	are in H.P.

50. Match the following :

Column-I		Column-II	
(A)	If three unequal numbers a, b, c are in A.P. and $b-a, c-b, a$ are in GP., then $\frac{a^3+b^3+c^3}{3abc}$ is equal to	(p)	4
(B)	Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers, then $\frac{y^3+z^3}{xyz}$ is equal to	(q)	1
(C)	If $a_1, a_2, a_3, ..., a_{50}$ are 50 distinct numbers in A.P. and $a_1^2 - a_2^2 + a_3^2 .... - a_{50}^2 = \left(\frac{5}{7}\right)^n (a_1^2 - a_{50}^2)$, $(n \in N)$ then $n =$	(r)	2

SECTION-V

Integer Answer Type

51. The value of $xyz = 55$ or $\frac{343}{55}$ according as the series a, x, y, z, b form an A.P or H.P respectively, where a and b are positive natural numbers. Find the sum $a + b$.

52. If the sum of the n terms of the series $1^3 + 3\cdot2^2 + 3^3 + 3\cdot4^2 + 5^3 + 3\cdot6^2 + ...$, where n is an even number, is given by $\frac{n}{k}(n^3 + an^2 + bn + c)$ then $b - a + c - k$ is

53. Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ where $f(x) = 2^x$

54. Let $S = \sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}} + \sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}} + ... + \sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}$

then find $|2000(S - 2000)|$.

55. Given a, b, c are positive integers forming an increasing G.P., $b - a$ is a perfect square of a natural number, and $\log_6 a + \log_6 b + \log_6 c = 6$. Find the value of $(a + b + c)/111$.

56. Find $\frac{S}{4}$, where S represent the sum to infinity of a decreasing G.P. with the common ratio x such that $|x| < 1; x \neq 0$. The ratio of the fourth term to the second term is $\frac{1}{16}$ and the ratio of third term to the square of the second term is 1/9.

57. If $(1 - P)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - P^6$, $P \neq 1$, then find the value of P/x.

58. If $(1^2 - a_1) + (2^2 - a_2) + (3^2 - a_3) + ... + (n^2 - a_n) = \frac{1}{3}n(n^2 - 1)$, then find the value of a_7.

59. If $S_n = 1\cdot n + 2\cdot(n - 1) + 3\cdot(n - 2) + ... + n\cdot1$ and $S_{25} = 325\lambda$ then λ is

60. Three numbers, the third of which is 4 form a decreasing G.P. If the last term is replaced by 3, the three numbers form an A.P, then the first number of the G.P. is

SOLUTIONS

1. (b) : $a_n = \frac{(\log_e 10)^n}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}$

$= \frac{(\log_e 10)^n}{n!}[2^n] = \frac{(2\log_e 10)^n}{n!}$

Thus, $a_0 + a_1 + a_2 + ...$ upto infinity is

$= \sum_{n=0}^{\infty}\frac{(2\log_e 10)^n}{n!} = e^{2\log_e 10} = 100$

2. (c) : $a_1 = 1$

$\Rightarrow a_2 = 1 - \frac{1}{2}$

$\Rightarrow a_3 = 1 - \frac{1}{2} + \frac{1}{3}$

.....

.....

$a_\infty = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln 2$

3. (d) : $\frac{1}{x_{k+1}} = \frac{1}{x_k(x_k+1)} = \frac{1}{x_k} - \frac{1}{x_k+1}$

$\Rightarrow \frac{1}{x_k+1} = \frac{1}{x_k} - \frac{1}{x_{k+1}}$

$\therefore \frac{1}{x_1+1} + \frac{1}{x_2+1} + ... + \frac{1}{x_{100}+1} = \frac{1}{x_1} - \frac{1}{x_{101}}$

As $0 < \frac{1}{x_{101}} < 1$

$\therefore \left[\frac{1}{x_1+1} + \frac{1}{x_2+1} + ... + \frac{1}{x_{100}+1}\right] = 1$

4. (c) : We have $t_n = S_n - S_{n-1}$ $\forall$ $n \geq 2$

$t_n = \frac{1}{6}[2(n^3 - (n-1)^3) + 9(n^2 - (n-1)^2) + 13(n - n + 1)]$

$= \frac{1}{6}[6n^2 - 6n + 2 + 9(2n-1) + 13]$

$= \frac{1}{6}(6n^2 + 12n + 6) = (n+1)^2$

$\therefore \sum_{r=1}^{n}\sqrt{t_r} = \sum_{r=1}^{n}(r+1) = \frac{1}{2}(n+1)(n+2) - 1 = \frac{1}{2}n(n+3)$

5. (c) : $a_1\ a_2\ ....\ a_n = b_n\frac{a_1a_2...a_n}{b_n}$

$= a_nb_n\frac{(a_1a_2...a_{n-1})}{b_n}$

$= \left(x^{\frac{1}{2^{n-1}}} - y^{\frac{1}{2^{n-1}}}\right)\frac{(a_1a_2...a_{n-1})}{b_n}$

$$= a_{n-1}b_{n-1}\frac{(a_1a_2...a_{n-2})}{b_n}$$

$$= \frac{a_1b_1}{b_n} = \frac{(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{b_n} = \frac{x-y}{b_n}$$

6. (c) : $T_r = \dfrac{4r+1}{5^r r(r-1)}, r \geq 2$

$$= \frac{5r-(r-1)}{5^r r(r-1)} = \frac{1}{5^{r-1}(r-1)} - \frac{1}{5^r r}$$

$$\sum_{r=2}^{\infty} T_r = \left(\left(\frac{1}{5^1\cdot 1} - \frac{1}{5^2\cdot 2}\right) + \left(\frac{1}{5^2\cdot 2} - \frac{1}{5^3\cdot 3}\right) + \left(\frac{1}{5^3\cdot 3} - \frac{1}{5^4\cdot 4}\right) + ... \infty\right)$$

$$= \frac{1}{5}$$

7. (c) : $S_1 = \dfrac{T_1^2 + 3T_1 - 2}{3} = T_1 \Rightarrow T_1^2 = 2$

$$S_2 - S_1 = \frac{T_2^2 - T_1^2 + 3(T_2 - T_1)}{3} = T_2$$

$\Rightarrow$ $T_2^2 - T_1^2 + 3(T_2 - T_1) = 3T_2$

$\Rightarrow$ $T_2^2 = T_1^2 + 3T_1 = 2 + 3\sqrt{2}$

8. (d) : a, b, c are in A.P. $\Rightarrow 2b = a + c$ ...(i)

$(b - a), (c - b), a$ are in G.P. $\Rightarrow (c - b)^2 = a(b - a)$ ...(ii)

From (i) and (ii), $\dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{3}$

9. (a) : $t_{n+1} = \dfrac{t_n \times (2n+1)}{(n+1)} \times \dfrac{1}{2}$

$\Rightarrow$ $(2n + 2)t_{n+1} = (2n + 1)t_n$

$(2n + 3)t_{n+1} - (2n + 1)t_n = t_{n+1}$

Put $n = 1$, $5t_2 - 3t_1 = t_2$

$n = 2$, $7t_3 - 5t_2 = t_3$

- - - - - - - - - - - - - - - - - -

$(2n + 1)t_n - (2n - 1)t_{n-1} = t_n$

$\Rightarrow$ $(2n + 1)t_n - 2t_1 = S$

$\Rightarrow$ $S = \dfrac{1\cdot 3\cdot 5.....(2n-1)(2n+1)}{\lfloor n \times 2^n} - 1$

10. (b) : Given $\dfrac{S_3}{S_6} = \dfrac{64}{91} = \dfrac{a(r^3-1)}{a(r^6-1)}$

$\Rightarrow$ $\dfrac{(r^3-1)}{(r^3+1)(r^3-1)} = \dfrac{64}{91}$

$\Rightarrow$ $r^3 = \dfrac{27}{64}$ $\therefore$ $r = \dfrac{3}{4}$

11. (b) : $S_n = 3 + 5 + 9 + 17 + 33 + ...$

$= (2 + 1) + (2^2 + 1) + (2^3 + 1) + (2^4 + 1) + ...$

$= (2 + 2^2 + 2^3 + 2^4 + ... n \text{ terms}) + n$

$= 2(2^n - 1) + n = 2^{n+1} + n - 2$

12. (b) : Given, $a + b = 2A$

And $a, p, q, b \in$ G.P.

$\therefore$ $p^2 = aq$ and $q^2 = pb$

$\Rightarrow$ $p^3 = apq$ and $q^3 = bpq$

By adding, we get

$p^3 + q^3 = apq + bpq$

$= pq(a + b) = 2Apq$

13. (b) : As the number of terms are odd (7), let r, be the common ratio.

So terms can be taken as $\dfrac{a}{r^3}, \dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2, ar^3$

$\therefore$ Product of the terms $= a^7 = 3^7$ as $(t_4 = a = 3)$

14. (c) : If n is a positive number, then

$n = p_1^{k_1} \cdot p_2^{k_2} ..... p_r^{k_r}$

(where $p_1, p_2, p_3, ..., p_r$ are prime numbers) then number of divisors of n are $(k_1 + 1)(k_2 + 1) ... (k_r + 1)$

$\therefore$ Number of prime factors of 6912 are $2^8 \cdot 3^3$

So no. of divisors $= 9 \times 4 = 36$

Prime factors of 52,488 are $3^8 \times 2^3$

$\therefore$ No. of divisors $= 9 \times 4 = 36$

Prime factors of 32,000 are $5^3 \times 2^8$

$\therefore$ No. of divisors $= 9 \times 4 = 36$

Now each number having same number of divisors *i.e.* 36, 36, 36

Each and every term is constant & constant sequence is always in A.P., G.P. and H.P. as common difference is 0 and common ratio is 1.

15. (c) : Given 1, $\log_{9^2}(3^x + 48), \log_9(3^x - 8/3) \in$ A.P.

$\Rightarrow$ $\log_9 9, \dfrac{1}{2}\log_9(3^x + 48), \log_9\left(3^x - \dfrac{8}{3}\right) \in$ A.P.

$\Rightarrow$ $9, (3^x + 48)^{1/2}, 3^x - 8/3 \in$ G.P. (by concept)

$\therefore$ $3^x + 48 = 9(3^x - 8/3)$

$\Rightarrow$ $8 \cdot 3^x = 72$

$\Rightarrow$ $3^x = 9 \Rightarrow 3^x = 3^2 \Rightarrow x = 2$

16. (b) : Given $a, b, c \in$ H.P.

So, $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c} \in$ A.P.

$\Rightarrow$ $\dfrac{a+b+c}{a}, \dfrac{b+c+a}{b}, \dfrac{c+a+b}{c} \in$ A.P.

By using concept if $a, b, c \in$ A.P.

Then their reciprocals are in H.P.

17. (b) : $\dfrac{a+b}{2} = 2\sqrt{ab} \Rightarrow a + b - 4\sqrt{ab} = 0$

$\Rightarrow$ $\dfrac{a}{b} + 1 - 4\sqrt{\dfrac{a}{b}} = 0$ (Dividing by b)

or $\left(\sqrt{\dfrac{a}{b}}\right)^2 - 4\sqrt{\dfrac{a}{b}} + 1 = 0$

$\therefore \quad \sqrt{\frac{a}{b}} = \frac{4 \pm 2\sqrt{3}}{2} = (2 \pm \sqrt{3})$

$\Rightarrow \quad \frac{a}{b} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$

18. (d) : Let m^{th} term of first A.P. be equal to the n^{th} term of the second A.P. then

2 , 5 , 8 , …., 50 terms (series 1)

3 , 5 ,7 , …., 60 terms (series 2)

Common series 5, 11, 17, ….., 119

40^{th} term of series 1 = 59^{th} term of series 2 = 119

= last term of common series

$\Rightarrow \quad a_n = 5 + (n - 1)d \quad \Rightarrow \quad 119 + 1 = 6n$

$\Rightarrow \quad n = 20$

$\therefore$ Number of common terms is 20.

19. (c) : Given expression is equal to

$$\frac{(a+b)}{c} + \frac{(b+c)}{a} + \frac{(c+a)}{b}$$

or $\quad \frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b}$

Using A.M. $\geq$ G.M.

$$\frac{\frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b}}{6} \geq \sqrt[6]{\frac{a}{c}\frac{b}{c}\frac{b}{a}\frac{c}{a}\frac{c}{b}\frac{a}{b}}$$

or $\quad \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6$

20. (c) : $S = ab + [ab + (a + b) + 1] + [ab + 2(a + b) + 2^2]$

$+ \ldots. + [ab + (n - 1)(a + b) + (n - 1)^2]$

$$= nab + (a+b)\sum_{r=1}^{n-1} r + \sum_{r=1}^{n-1} r^2$$

$$= nab + (a+b)\frac{n(n-1)}{2} + \frac{(n-1)(n)(2n-1)}{6}$$

$$= \frac{n}{6}[1 + (n-1)\{1 + 2n - 1\}]$$

$$= \frac{n}{6}[1 + 2n(n-1)] = \frac{n}{6}(1 - 2n + 2n^2)$$

21. (a) : $\log(a + c) + \log(b + c) = 2\log(a + b)$

$\Rightarrow \quad (a + c)(b + c) = (a + b)^2$

$\Rightarrow \quad ab + c(a + b) + c^2 = (a + b)^2$

$\Rightarrow \quad 2ab + 2c(a + b) + 2c^2 = 2(a + b)^2 \qquad \ldots(i)$

Also, $c = \frac{2ab}{a+b} \quad \Rightarrow \quad 2ab = c(a+b) \qquad \ldots(ii)$

From (i) and(ii),

$c(a + b) + 2c(a + b) + 2c^2 = 2(a + b)^2$

$2(a + b)^2 - 3c(a + b) - 2c^2 = 0$

$$\therefore \quad a + b = \frac{3c \pm \sqrt{9c^2 + 16c^2}}{4} = \frac{3c \pm 5c}{4} = 2c \text{ or } -\frac{c}{2}$$

$\therefore \quad a + b = 2c. \quad (\because a, b, c > 0)$

22. (c) : We have, $2b = a + c$ and a, p, b, q, c are in A.P.

$\Rightarrow \quad p = \frac{a+b}{2}, q = \frac{b+c}{2}$

Again, $p' = \sqrt{ab}$ and $q' = \sqrt{bc}$

$$\therefore \quad p^2 - q^2 = \frac{(a+b)^2 - (b+c)^2}{4}$$

$$= \frac{(a-c)(a+c+2b)}{4} = (a-c)b = p'^2 - q'^2$$

23. (d) : $\log_6(abc) = 6 \quad \Rightarrow \quad (abc) = 6^6$

Let $a = b/r$ and $c = br$

$\Rightarrow \quad b = 36$ and $a = \frac{36}{r} \quad \Rightarrow \quad r = 2, 3, 4, 6, 9, 12, 18$

Also $b - a = 36\left(1 - \frac{1}{r}\right)$ is a perfect cube.

$\therefore \quad r = 4$

$\Rightarrow \quad a + b + c = 36 + 9 + 144 = 189$

24. (b) : $S = \frac{a(1-r^n)}{1-r}, \quad P = a^n \cdot r^{\frac{n(n-1)}{2}}$

$$R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ldots. \text{ to } n \text{ terms} = \frac{1-r^n}{a(1-r)r^{n-1}}$$

$$S^n = R^n P^k \quad \Rightarrow \quad \left(\frac{S}{R}\right)^n = P^k$$

$\Rightarrow \quad (a^2 r^{n-1})^n = P^k$

$\Rightarrow \quad P^2 = P^k \quad \Rightarrow \quad k = 2$

25. (a) : Since $A_1, A_2, A_3, \ldots., A_{11}$ be 11 A.M.s between 28 and 10.

$\therefore \quad 28, A_1, A_2, \ldots, A_{11}, 10$ are in A.P.

Let 'd' be the common difference of A.P.

Also the number of terms = 13.

$10 = T_{13} = T_1 + 12d = 28 + 12d$

$$\therefore \quad d = \frac{10-28}{12} = -\frac{18}{12} = -\frac{3}{2}$$

$\therefore$ Number of integral A.M's is 5.

26. (a) : Given a, b, c are in H.P., so $b = \frac{2ac}{(a+c)}$

$$\therefore \quad \frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{\frac{2ac}{(a+c)} - a} + \frac{1}{\frac{2ac}{(a+c)} - c}$$

$$= (a+c)\left\{\frac{1}{a(c-a)} + \frac{1}{c(a-c)}\right\}$$

$$= \frac{(a+c)}{(a-c)}\left\{-\frac{1}{a} + \frac{1}{c}\right\}$$

$$= \frac{(c+a)(a-c)}{ac(a-c)} \quad \Rightarrow \quad \frac{(a+c)}{ac} = \frac{2}{b}$$

27. (d) : $a_1 + a_3 + a_5 + \ldots. + a_{199} = \beta$

$a_2 + a_4 + a_6 + \ldots + a_{200} = \alpha$

$a_2 - a_1 + a_4 - a_3 + a_6 - a_5 + ... + a_{200} - a_{199} = \alpha - \beta$

$\Rightarrow$ $d + d + d + ...$ upto 100 times $= \alpha - \beta$

$\Rightarrow$ $d = \dfrac{\alpha - \beta}{100}$

28. (b) : a, b, c, d are in G.P., let they are a, ar, ar^2, ar^3

Now $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= a^2 \times a^2[1 + r^2 + r^4][r^2 + r^4 + r^6]$

$= a^4r^2[1 + r^2 + r^4]^2$

$= [a^2r[1 + r^2 + r^4]]^2$

$= (ab + bc + cd)^2$

29. (c) : a_1, a_2, a_3, a_4, a_5 are in H.P.

$\Rightarrow$ $a_2 = \dfrac{2a_1a_3}{a_1 + a_3} \Rightarrow 2a_1a_3 = a_2a_1 + a_3a_2$

$a_4 = \dfrac{2a_3a_5}{a_3 + a_5} \Rightarrow 2a_3a_5 = a_3a_4 + a_5a_4$

$\Rightarrow$ $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 = 2a_1a_3 + 2a_3a_5$ ...(i)

$a_3 = \dfrac{2(a_1a_5)}{a_1 + a_5} \Rightarrow a_1a_3 + a_5a_3 = 2a_1a_5$ ...(ii)

Using (i) & (ii)

$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 = 2(2a_1a_5) = 4a_1a_5$

30. (c, d) : Let $b = a + p, c = a + 2p, d = a + 3p$

$$\frac{\frac{1}{a}+\frac{1}{d}}{\frac{1}{b}+\frac{1}{c}} = \frac{\frac{1}{a}+\frac{1}{a+3p}}{\frac{1}{a+p}+\frac{1}{a+2p}} = \frac{(a+p)(a+2p)}{a(a+3p)}$$

$$= \frac{a^2+3ap+2p^2}{a^2+3ap} > 1$$

$\therefore$ $\dfrac{1}{a} + \dfrac{1}{d} > \dfrac{1}{b} + \dfrac{1}{c}$

Now $\left(\dfrac{1}{b} + \dfrac{1}{c}\right)(a + d) = \left(\dfrac{1}{a+p} + \dfrac{1}{a+2p}\right)(a + a + 3p)$

$$= \frac{(2a+3p)^2}{a^2+3ap+2p^2} = 4 + \frac{p^2}{a^2+3ap+2p^2} > 4$$

31. (b, c) : Let A.P. be $A, A + d, A + 2d, ...$ then

$t_{2n-1} = A + (2n - 2)d = x$ (say), then

$(n-1)d = \dfrac{x - A}{2}$

$\therefore$ $a = t_n = A + (n - 1)d = A + \dfrac{x - A}{2} = \dfrac{A + x}{2}$

Let G.P. be $A, AR, AR^2, ...$ then

$t_{2n-1} = AR^{2n-2} = x \Rightarrow R^{n-1} = \left(\dfrac{x}{A}\right)^{1/2}$

$\therefore$ $b = t_n = AR^{n-1} = A\left(\dfrac{x}{A}\right)^{1/2} \Rightarrow \sqrt{Ax}$

Let H.P. be $A, \dfrac{1}{\frac{1}{A}+D}, \dfrac{1}{\frac{1}{A}+2D}$ ..... then

$t_{2n-1} = \dfrac{1}{\frac{1}{A}+(2n-2)D} = x$, then

$(n-1)D = \dfrac{1}{2}\left(\dfrac{1}{x} - \dfrac{1}{A}\right)$

$\therefore$ $c = t_n = \dfrac{1}{\frac{1}{A}+(n-1)D} = \dfrac{1}{\frac{1}{A}+\frac{1}{2}\left(\frac{1}{x}-\frac{1}{A}\right)}$

$= \dfrac{1}{\frac{1}{2}\left(\frac{1}{x}+\frac{1}{A}\right)}$

Clearly a, b, and c are A.M., G.M. and H.M. between the numbers, x and A respectively.

Hence $a \geq b \geq c$, also $b^2 = ac$.

32. (a, b, c, d) : Let a, ar, ar^2, ar^3 are the first four terms of the G.P.

$\therefore$ $a^4r^6 = 4$ and $a^2r^4 = 1$

$\Rightarrow$ $r^2 = \dfrac{1}{4} \Rightarrow r = \pm\dfrac{1}{2}$ and $a = \pm 4$

$\therefore$ Sum of infinite G.P. $= \dfrac{a}{1-r} = 8, -8, \dfrac{8}{3}, -\dfrac{8}{3}$

33. (a, b, c, d) : $a_1 + a_4 + a_7 + ... + a_{16} = 147$

$\Rightarrow$ $3(a_1 + a_{16}) = 147 \Rightarrow a_1 + a_{16} = 49$

Again, $a_1 + a_4 + a_7 + a_{10} + ... + a_{16}$

$= a_1 + a_1 + 3d + a_1 + 6d + ... + a_1 + 15d$

$= 6a_1 + 45d = 147$

$\Rightarrow$ $2a_1 + 15d = 49$

$a_1 + a_6 + a_{11} + a_{16} = a_1 + a_1 + 5d + a_1 + 10d + a_1 + 15d$

$= 4a_1 + 30d = 2(2a_1 + 15d)$

$= 2(49) = 98$

Now using A.M. $\geq$ G.M.

$\Rightarrow$ $\dfrac{a_1 + a_2 + ... + a_{16}}{16} \geq (a_1a_2a_3.... a_{16})^{1/16}$

$\dfrac{8(a_1 + a_{16})}{16} \geq (a_1a_2a_3 .... a_{16})^{1/16}$

$\left(\dfrac{49}{2}\right)^{16} \geq a_1a_2a_3 .... a_{16}$

34. (a, c, d) : $T_r = \dfrac{r(\sqrt{r+1}) - (r+1)\sqrt{r}}{r^2(r+1) - (r+1)^2 r}$

$= \dfrac{r\sqrt{r+1} - (r+1)\sqrt{r}}{-r^2 - r} = \dfrac{(r+1)\sqrt{r}}{r(r+1)} - \dfrac{r\sqrt{r+1}}{r(r+1)}$

$= \dfrac{1}{\sqrt{r}} - \dfrac{1}{\sqrt{r+1}}$

$$\Rightarrow \sum_{r=1}^{99} T_r = \frac{1}{1} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} .... - \frac{1}{\sqrt{100}}$$

$$= 1 - \frac{1}{\sqrt{100}} = \frac{9}{10}$$

35. (a, d) : $S_{(1)} + S_{(2)} + ... + S_{(n-1)}$

$S_{(1)} : 1$

$S_{(2)} : 1 + \frac{1}{2}$

$S_{(3)} : 1 + \frac{1}{2} + \frac{1}{3}$

.....

.....

$S_{(n-1)} : 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1}$

Adding vertically :

$$= (n-1) + \frac{(n-2)}{2} + \frac{(n-3)}{3} + ... + \left(\frac{n-(n-1)}{(n-1)}\right)$$

$$= n\left[1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1}\right]$$

$- [1 + 1 + 1 + \ldots \text{ upto } (n-1) \text{ times}]$

$= nS_{(n-1)} - (n-1) = nS_n - n$

36. (a, b) : We have

$$\frac{1}{2}(2 + a_1) \geq \sqrt{2a_1}$$

$$\frac{1}{2}(2 + a_2) \geq \sqrt{2a_2}$$

.....

$$\frac{1}{2}(2 + a_n) \geq \sqrt{2a_n}$$

Multiplying above inequalities, we get

$$\frac{1}{2^n}(2 + a_1)(2 + a_2) .... (2 + a_n)$$

$$\geq 2^{n/2} \sqrt{a_1 a_2 ..... a_n} = 2^{n/2}$$

$\Rightarrow (2 + a_1)(2 + a_2) .... (2 + a_n) \geq 2^{3n/2}$

As all $a_i \neq 2$, thus we have strict inequality in the above inequality.

37. (a, b, c) : T_p of A.P. $= \frac{1}{q(p+q)} = A + (p-1)D$ ...(i)

T_q of A.P. $= \frac{1}{p(p+q)} = A + (q-1)D$ ...(ii)

$$\frac{1}{T_{p+q}} = A + (p+q-1)D$$

and $\frac{1}{T_{pq}} = A + (pq-1)D$

Now, solving (i) and (ii), we get

$$A = D = \frac{1}{pq(q+p)}$$

$$\therefore \frac{1}{T_{p+q}} = A + (p+q-1)D = (p+q)D = \frac{1}{pq}$$

and $\frac{1}{T_{pq}} = A + (pq - 1)D = pqD = \frac{1}{p+q}$

$\Rightarrow T_{p+q} = pq$ and $T_{pq} = p + q$

Also, $\because pq > p + q$

i.e., $T_{p+q} > T_{pq}$

38. (a, c) : $b_2 = b_1 r$, $b_3 = b_1 r^2$

$b_3 > 4b_2 - 3b_1$

$\Rightarrow b_1 r^2 > 4b_1 r - 3b_1 \Rightarrow r^2 > 4r - 3$

$\Rightarrow r^2 - 4r + 3 > 0 \Rightarrow (r-1)(r-3) > 0$

$\Rightarrow r > 3$ or $r < 1$

39. (a, b, c, d) : a, b, c are in H.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P.

$\Rightarrow \frac{b+c}{a} - 1, \frac{c+a}{b} - 1, \frac{a+b}{c} - 1$ are in A.P.

[subtracting 1 from each term]

Thus $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

And $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are also in H.P.

Also $b = \frac{2ac}{(a+c)}$, so $\frac{1}{b-a} + \frac{1}{b-c} = \frac{2b-(a+c)}{(b-a)(b-c)}$

$$= \frac{2b-(a+c)}{b^2 - b(a+c) + ac} = \frac{2b - 2ac/b}{b^2 - b(a+c) + ac}$$

$$= \frac{2}{b} \cdot \frac{b^2 - ac}{b^2 - ac} = \frac{2}{b}$$

Lastly, $\left(a - \frac{b}{2}\right)\left(c - \frac{b}{2}\right) = ac - \frac{b}{2}(a+c) + \frac{b^2}{4}$

$$= ac - \frac{b}{2} \cdot \frac{2ac}{b} + \frac{b^2}{4} = \frac{b^2}{4}$$

$\Rightarrow a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in G.P.

(40 - 42) :

40. (c) : $4^{\sin^{-1} x} + 4^{\cos^{-1} x} \geq 2\sqrt{4^{\sin^{-1} x + \cos^{-1} x}}$

$= 2 \cdot 2^{\pi/2}$

Equality holds if and only if $4^{\sin^{-1} x} = 4^{\cos^{-1} x}$

i.e. $x = \frac{1}{\sqrt{2}}$

41. (d) : Options (a), (b), (c) are correct. Only (d) option is not correct.

i.e. $x^2+\frac{1}{x^2}+4-3\left|x+\frac{1}{x}\right|=u^2-3u+2$

$= (u - 1)(u - 2) \geq 0$

where $u=\left|x+\frac{1}{x}\right|$. Then $u \geq 2$

42. (d) : If x_1, x_2, x_3, x_4 denote the roots of the given equation $\sum x_1 = 4$, $x_1x_2x_3x_4 = 1$

$\Rightarrow \frac{\sum x_1}{4}=\sqrt[4]{x_1x_2x_3x_4}$

Hence $x_1 = x_2 = x_3 = x_4 = 1$

$\Rightarrow x^4 - 4x^3 + ax^2 - bx + 1 = (x - 1)^4$

$= x^4 - 4x^3 + 6x^2 - 4x + 1$

Thus $a = 6$, $b = 4$.

Then $a + b = 10$.

(43 - 45) :

43. (d) : $T_n = an^3 + bn^2 + cn + d$

$T_1 = a + b + c + d = 1$

$T_2 = 8a + 4b + 2c + d = 6$

$\Rightarrow 6a + 2b - d = 4$

44. (a) : $T_n=\frac{5}{4}3^{n-1}-\frac{5n}{2}+\frac{9}{4}$

$$S_{50}=\frac{5}{4}(1+3+...+3^{49})-\frac{5}{2}(1+2+...+50)+50\cdot\frac{9}{4}$$

$$=\frac{5}{4}\left(\frac{3^{50}-1}{2}\right)-\frac{5}{2}\cdot\frac{50\cdot51}{2}+\frac{450}{4}$$

$$=\frac{5}{8}(3^{50}-1)-\frac{125\cdot51}{2}+\frac{450}{4}$$

$$=\frac{5}{8}(3^{50}-1)-3075$$

45. (a) : $\sum_{r=1}^{n} r(n-r+1)=\sum_{r=1}^{n}(n+1)r-\sum_{r=1}^{n}r^2$

$$=(n+1)\sum n-\sum n^2$$

$$=\frac{(n+1)^2n}{2}-\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n(n+1)}{6}(3n+3-2n-1)=\frac{n(n+1)(n+2)}{6}$$

(46 - 48) :

46. (a) **47. (b)** **48. (d)**

1st $(2n + 1)$ terms of A.P. are $A, A + 2, ..., A + 4n$.

Last $(2n + 1)$ terms of G.P. are

$$(A+4n), (A+4n)\frac{1}{2}, ...,(A+4n)\frac{1}{2^{2n}}$$

$$\Rightarrow A+2n=\frac{A+4n}{2^n} \Rightarrow A=\frac{4n-2n\cdot2^n}{2^n-1}$$

Middle term of sequence $= T_{2n+1}$

$$= A+4n=\frac{n\,2^{n+1}}{2^n-1}$$

Middle term of G.P. $= T_{n+1} =\frac{2n\cdot2^n}{2^n-1}\times\frac{1}{2^n}=\frac{2n}{2^n-1}$

49. A - r; B - q; C - q; D - p, q, r, s

(A) $A_p = a + (p - 1)d$ ...(i)

$A_q = a + (q - 1)d$ ...(ii)

$A_r = a + (r - 1)d$ ...(iii)

$A_s = a + (s - 1)d$ ...(iv)

$A_q = kA_p$

$A_r = k^2A_p$

$A_s = k^3A_p$ ($\because A_p, A_q, A_r, A_s$ are in G.P.)

$(p-q)=\frac{A_p-A_q}{d}=A_p\frac{(1-k)}{d}$ (from (i) and (ii))

$(q-r)=A_pk\frac{(1-k)}{d}$ (from (ii) and (iii))

$(r-s)=A_pk^2\frac{(1-k)}{d}$ (form (iii) and (iv))

$\Rightarrow$ $p - q, q - r, r - s$ are in G.P.

(B) $\ln x, \ln y, \ln z$ are in G.P.

$\Rightarrow$ $\ln(\ln x), \ln(\ln y), \ln(\ln z)$ are in A.P

$\Rightarrow$ $2x + \ln(\ln x), 3x + \ln(\ln y), 4x + \ln(\ln z)$ are in A.P.

(C) $n!$, $3 \times n!$ and $(n + 1)!$ are in G.P.

$\Rightarrow 9(n!)^2 = n!(n + 1)!$

$\Rightarrow (n + 1) = 9 \Rightarrow n = 8 \quad \therefore n! = 8!$

$5 \times n! = 5 \times 8!$

$(n + 1)! = 9!$

$\Rightarrow 9! + 8! = 2(5 \times 8!)$

$\Rightarrow$ $n!$, $5 \times n!$ and $(n + 1)!$ are in A.P.

(D) $\frac{(b-c)^2+(a-b)^2+(c-a)^2}{3}$

$$=\frac{(b-c-2a)^2+(c+a-2b)^2+(a+b-2c)^2}{3}$$

$\Rightarrow (b + c - 2a)^2 - (b - c)^2 + (c + a - 2b)^2$

$- (c - a)^2 + (a + b - 2c)^2 - (a - b)^2 = 0$

$\Rightarrow a = b = c$

50. A – r, B – r , C – r

(A) $(b - a) = (c - b)$ and $(c - b)^2 = a(b - a)$

$\Rightarrow (b - a)^2 = a(b - a) \Rightarrow b = 2a, c = 3a$

$\therefore a : b : c = 1 : 2 : 3$

(B) $x=\frac{a+b}{2}, b=ar^3 \Rightarrow r=\left(\frac{b}{a}\right)^{1/3}$

$\Rightarrow \frac{y^3+z^3}{xyz}=\frac{a+b}{\frac{a+b}{2}}=2$

(C) $a_1^2 - a_2^2 + a_3^2 - .... - a_{50}^2$

$= (a_1 + a_2)(a_1 - a_2) + (a_3 + a_4)(a_3 - a_4) + ...$
$+ (a_{49} + a_{50})(a_{49} - a_{50})$

$= -d[a_1 + a_2 + ... + a_{50}]$

$= -\frac{25}{49}(a_{50} - a_1)(a_{50} + a_1)$

$= \left(\frac{25}{49}\right)(a_1^2 - a_{50}^2)$

51. (8) : If a, x, y, z, b to are in A.P then the common difference d of the A.P. is given by

$$b = a + 4d \Rightarrow d = \frac{b-a}{4}$$

$$\therefore \quad x = a + d = a + \frac{b-a}{4} = \frac{b+3a}{4}$$

$$y = a + 2d = a + \frac{b-a}{2} = \frac{a+b}{2}$$

$$z = a + 3d = a + 3\left(\frac{b-a}{4}\right) = \frac{a+3b}{4}$$

$$\therefore \quad xyz = \frac{b+3a}{4} \times \frac{a+b}{2} \times \frac{a+3b}{4}$$

$$\Rightarrow \quad 55 = \frac{(3a+b)(a+b)(a+3b)}{32}$$

$\Rightarrow$ $(3a + b)(a + b)(a + 3b) = 55 \times 32$ ...(i)

When a, x, y, z, b are in H.P. then

$\frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in A.P.

Let D be the common difference of this A.P. Then

$$\frac{1}{b} = \frac{1}{a} + 4D \Rightarrow D = \frac{a-b}{4ab}$$

$$\therefore \quad \frac{1}{x} = \frac{1}{a} + D = \frac{1}{a} + \frac{a-b}{4ab} = \frac{3b+a}{4ab}$$

$$\frac{1}{y} = \frac{1}{a} + 2D = \frac{1}{a} + \frac{a-b}{2ab} = \frac{a+b}{2ab}$$

$$\frac{1}{z} = \frac{1}{a} + 3D = \frac{1}{a} + \frac{3(a-b)}{4ab} = \frac{3a+b}{4ab}$$

$$\therefore \quad \frac{1}{x} \cdot \frac{1}{y} \cdot \frac{1}{z} = \frac{(3a+b)(a+b)(a+3b)}{32a^3 b^3}$$

$$\Rightarrow \quad \frac{1}{xyz} = \frac{(3a+b)(a+b)(a+3b)}{32a^3 b^3} \text{ (using (i))}$$

$$\Rightarrow \quad \frac{55}{343} = \frac{55 \times 32}{32\, a^3 b^3}$$

$\Rightarrow$ $(ab)^3 = 7^3 \Rightarrow ab = 7$

$\Rightarrow$ $a = 1, b = 7$ or $a = 7, b = 1$

52. (6) : We have

$S = 1^3 + 3 \cdot 2^2 + 3^3 + 3 \cdot 4^2 + 5^3 + 3 \cdot 6^2 + ...$

$S = (1^3 + 3^3 + 5^3 + ...) + 3 \cdot (2^2 + 4^2 + 6^2 + ...)$

$S = (1^3 + 3^3 + 5^3 + ...) + 12(1^2 + 2^2 + 3^2 + ...)$

Where $S_1 = 1^3 + 3^3 + 5^3 + \ldots$ and $S_2 = 1^2 + 2^2 + 3^2 + ...$

Now case arise

When n is even, say $n = 2m$, $m \in N$

In this case both S_1 and S_2 contain m terms

$\therefore \quad S_1 = 1^3 + 3^3 + 5^3 + ... + (2m - 1)^3$

$$= \sum_{r=1}^{m} (2r-1)^3$$

$$= \sum_{r=1}^{m} (8r^3 - 12r^2 + 6r - 1)$$

$$= 8\sum_{r=1}^{m} r^3 - 12\sum_{r=1}^{m} r^2 + 6\sum_{r=1}^{m} r - \sum_{r=1}^{m} 1$$

$$= 8\left\{\frac{m(m+1)}{2}\right\}^2 - 12\left\{\frac{m(m+1)(2m+1)}{6}\right\} + \frac{6m(m+1)}{2} - m$$

$$= 8\left\{\frac{n(n+2)}{8}\right\}^2 - \frac{12}{6}\left\{\frac{n}{2}\left(\frac{n+2}{2}\right)(n+1)\right\} + 3\frac{n}{2}\left(\frac{n+2}{2}\right) - \frac{n}{2}$$

$$= \frac{n^2(n+2)^2}{8} - \frac{n(n+1)(n+2)}{2} + 3\frac{n(n+2)}{4} - \frac{n}{2}$$

$S_2 = 1^2 + 2^2 + 3^2 + ... + m^2$

$$= \frac{m(m+1)(2m+1)}{6} = \frac{n(n+2)(n+1)}{24}$$

$\therefore \quad S = S_1 + 12S_2$

$$= \frac{n^2(n+2)^2}{8} - \frac{n(n+1)(n+2)}{2} + \frac{3}{4}n(n+2) - \frac{n}{2} + \frac{n(n+1)(n+2)}{2}$$

$$= \frac{n^2(n+2)^2}{8} + \frac{3}{4}n(n+2) - \frac{n}{2}$$

$$= \frac{n}{8}(n^3 + 4n^2 + 10n + 8)$$

53. (3) : $f(x) = 2^x$ for all $x \in N$

$$\therefore \quad \sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$$

$$\Rightarrow \quad \sum_{k=1}^{n} 2^{a+k} = 16(2^n - 1)$$

$\Rightarrow \sum_{k=1}^{n} 2^a \cdot 2^k = 16(2^n - 1)$

$\Rightarrow 2^a\left(\sum_{k=1}^{n} 2^k\right) = 16(2^n - 1)$

$\Rightarrow 2^a(2 + 2^2 + ... + 2^n) = 16(2^n - 1)$

$\Rightarrow 2^a\left\{2\left(\frac{2^n - 1}{2 - 1}\right)\right\} = 16(2^n - 1)$

$\Rightarrow 2^{a+1}(2^n - 1) = 16(2^n - 1)$

$\Rightarrow 2^{a+1} = 2^4 \Rightarrow a + 1 = 4 \Rightarrow a = 3$

54. (1) : $t_r = \sqrt{1 + \frac{1}{r^2} + \frac{1}{(r+1)^2}}$

$= \sqrt{\frac{r^2 + (r+1)^2 + r^2(r+1)^2}{r^2(r+1)^2}}$

$= \sqrt{\frac{2r^2 + 2r + 1 + r^2(r^2 + 2r + 1)}{r^2(r+1)^2}}$

$= \sqrt{\frac{r^4 + 2r^3 + 3r^2 + 2r + 1}{r^2(r+1)^2}}$

$= \frac{r^2 + r + 1}{r(r+1)} = \frac{1}{r(r+1)} + 1 = 1 + \frac{1}{r} - \frac{1}{r+1}$

$S = 2000 - \frac{1}{2000}, |2000(S - 2000)| = 1$

55. (1) : a, b, c are in G.P.

$\Rightarrow b^2 = ac$

$\log_6 a + \log_6 b + \log_6 c = 6$

$abc = 6^6$

$b^3 = 6^6$

$b = 6^2 = 36$

$ac = 36 \times 36 = 2^4 \times 3^4$

$b - a = N^2$

$36 - a = N^2$

a is factor of $2^4\, 3^4$.

$a = 27$ is possible value.

$36 - 27 = 9 = (3)^2$

$\Rightarrow a = 27, b = 36, c = 48$

$\Rightarrow a + b + c = 111$

56. (3) : Let the series be $a, ax, ax^2, ax^3, ...$ given that $|x| < 1$ and $x \neq 0$

Also, $\frac{T_4}{T_2} = \frac{ax^3}{ax} = \frac{1}{16} \Rightarrow x^2 = \frac{1}{16}$

$\Rightarrow x = \pm\frac{1}{4}$

But since it is a decreasing G.P. $\Rightarrow x = \frac{1}{4}$

Also, $\frac{T_3}{T_2^2} = \frac{ax^2}{(ax)^2} = \frac{1}{9} \Rightarrow \frac{1}{a} = \frac{1}{9} \Rightarrow a = 9$

Now, $S = \frac{a}{1-r} = \frac{9}{1 - \frac{1}{4}} = \frac{9 \times 4}{3} = 12$

57. (3) : $(1 - P)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5)$
$= 1 - P^6$

$\Rightarrow (1 - P)\frac{1 - (3x)^6}{1 - 3x} = 1 - P^6$

which is possible only if $P = 3x$

$\therefore \frac{P}{x} = 3$

58. (7) : $(1^2 + 2^2 + ... + n^2) - (a_1 + a_2 + ... + a_n)$
$= \frac{1}{3}n(n^2 - 1)$ ...(i)

Replacing n by $(n - 1)$, then

$(1^2 + 2^2 + ... + (n - 1)^2) - (a_1 + a_2 + ... + a_{n-1})$
$= \frac{1}{3}(n-1)((n-1)^2 - 1)$ ...(ii)

Subtracting (ii) from (i), we get

$n^2 - a_n = n^2 - n$

$\Rightarrow a_n = n \Rightarrow a_7 = 7$

59. (9) : Let $T_r = r(n - r + 1) \Rightarrow T_r = nr - r^2 + r$

$S_n = \sum_{r=1}^{n} T_r = n\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 + \sum r$

$= \frac{n \times n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}\left[n - \frac{(2n+1)}{3} + 1\right]$

$= \frac{n(n+1)}{2}\left[\frac{3n - 2n - 1 + 3}{3}\right] = \frac{n(n+1)(n+2)}{6}$

$S_n = \frac{n(n+1)(n+2)}{6}, S_{25} = \frac{25 \times 26 \times 27}{6} = 25 \times 13 \times 9$

$S_{25} = 325\lambda \Rightarrow \lambda = 9$

60. (9) : Let a, ar, ar^2 be in G.P.

$2ar = a + 3 \Rightarrow a = \frac{3}{2r - 1}$ ...(i)

$ar^2 = 4$ ...(ii)

Solving (i) and (ii), we get $r = 2/3$, $a = 9$

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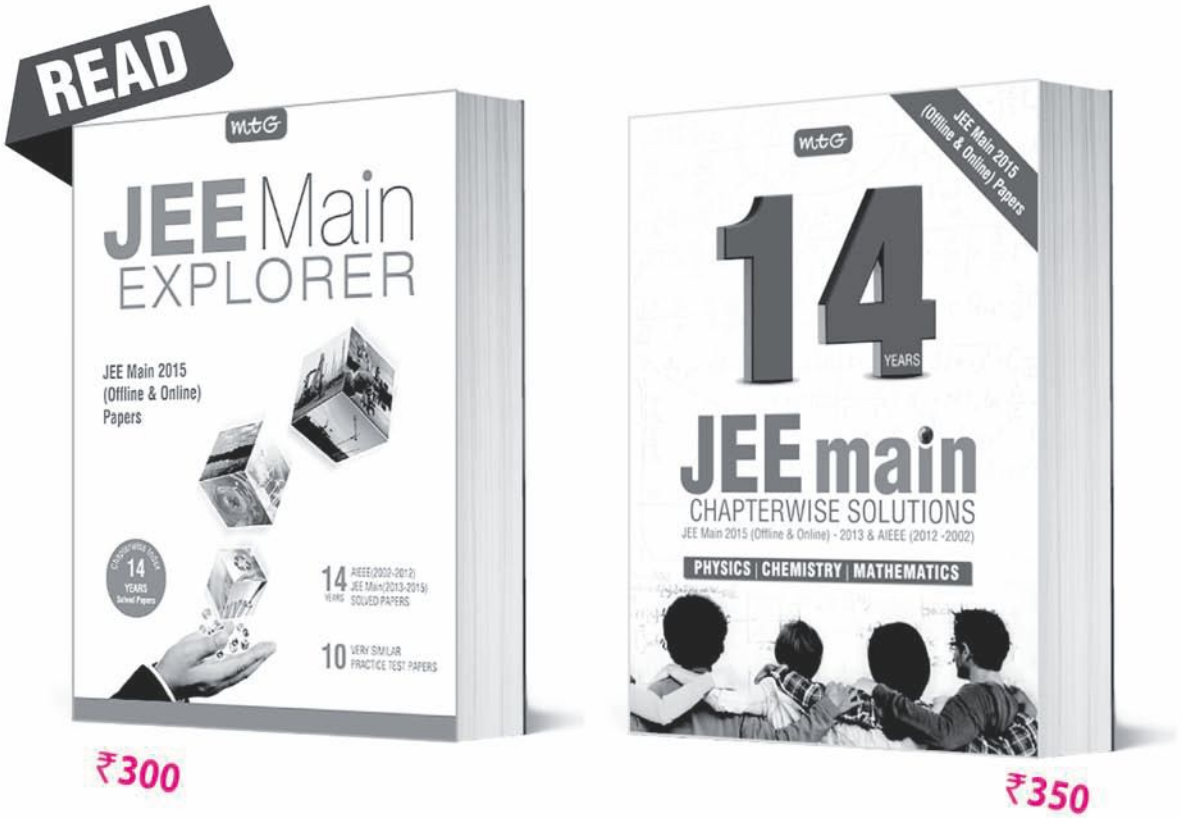

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CONCEPT BOOSTERS

Class XII

LIMITS AND DIFFERENTIABILITY

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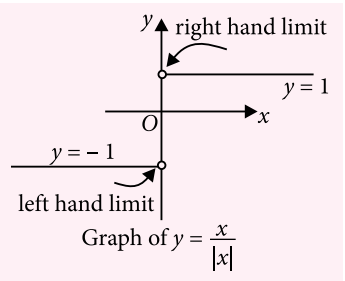
This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION OF LIMIT

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that $f(x)$ approaches the limit L as x approaches x_0 and write $\lim\limits_{x\to x_0} f(x)=L$, if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta$ and $|f(x) - L| < \varepsilon$.

Left Hand Limit

$\lim\limits_{x\to x_0^-} f(x)=L$, if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x satisfying $x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$ and we call it the left hand limit. *e.g.* $\lim\limits_{x\to 0}\dfrac{x}{|x|} = -1$



Right Hand Limit

$\lim\limits_{x\to x_0^+} f(x)=L$, if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x satisfying $x_0 < x < x_0 + \delta$ $\Rightarrow |f(x) - L| < \in$ and we call it the right hand limit *e.g.* $\lim\limits_{x\to 0^+}\dfrac{x}{|x|}=1$

- **Algebra of Limits**

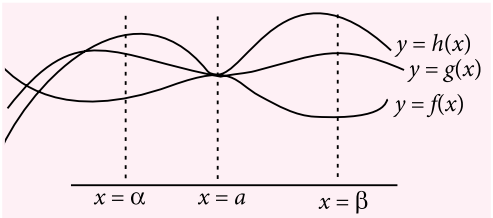
Let $\lim\limits_{x\to a} f(x)=l$ and $\lim\limits_{x\to a} g(x)=m$. If l and m are finite then :

- $\lim\limits_{x\to a}(f(x)\pm g(x))=l\pm m$
- $\lim\limits_{x\to a} f(x)\cdot g(x)=l\cdot m$
- $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\dfrac{l}{m}$, provided $m\neq 0$
- $\lim\limits_{x\to a} k\,f(x)=k\lim\limits_{x\to a} f(x)=kl$; where k is a constant
- $\lim\limits_{x\to a}[f(x)+k]=\lim\limits_{x\to a} f(x)+k$, where k is a constant
- If $f(x)\le g(x)$, for $x\in(a-h,a+h)-\{a\}$, for some $h>0$, then $\lim\limits_{x\to a} f(x)\le\lim\limits_{x\to a} g(x)$

- **Limit of a Composite Function**

$\lim\limits_{x\to a} f(g(x))=f\left(\lim\limits_{x\to a} g(x)\right)$; provided f is continuous at $x=\lim\limits_{x\to a} g(x)$

- **Squeeze play theorem (Sandwich Theorem)**



If $f(x)\le g(x)\le h(x)\ \forall\ x\in(\alpha,\beta)-\{a\}$ and $\lim\limits_{x\to a} f(x)=l=\lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x)=l$ where $a\in(\alpha,\beta)$.

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- **Few Indeterminate Forms are as Follows**

$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty,\ 0^0, \infty^0, \infty-\infty$ and 1^∞,

where zero does not stand for exactly zero but a quantity approaching towards zero similarly one does not stand for exactly one but a quantity approaching towards one.

- **Some Important Expansions**

(i) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

$= \lim_{n\to\infty} \sum_{r=1}^{n} (-1)^{r+1} \frac{x^r}{r}; -1 < x \le 1$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots\ldots$

$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{x^r}{r!}$

(iii) $a^x = 1 + (x\ln a) + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \ldots + \frac{(x\ln a)^n}{n!} + \ldots$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \lim_{n\to\infty} \sum_{r=0}^{n} (-1)^r \frac{x^{2r}}{2r!}$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots$

(vii) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots, -1 < x < 1$

(for rational or integral n, for irrational n, rule is not applicable)

SOME STANDARD RESULTS ON LIMITS

- **Limits of polynomial and rational functions**

(i) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(iii) $\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + \ldots.}{b_0x^n + b_1x^{n-1} + \ldots} = \begin{cases} \frac{a_0}{b_0}, & \text{when } m = n \\ 0, & \text{when } m < n \\ \infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} > 0 \\ -\infty, & \text{when } m > n \text{ and } \frac{a_0}{b_0} < 0 \end{cases}$

- **Limits of trigonometric functions**

(i) $\lim_{x\to 0} \sin x = \lim_{x\to 0} \tan x = 0$

(ii) $\lim_{x\to 0} \cos x = \lim_{x\to 0} \left(\frac{1}{\cos x}\right) = 1$

(iii) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{\sin^{-1} x}{x}$

$= \lim_{x\to 0} \frac{x}{\sin^{-1} x} = 1$

(where x is measured in radians)

(iv) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{x}{\tan x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$

$= \lim_{x\to 0} \frac{x}{\tan^{-1} x} = 1$

(where x is measured in radians)

(v) $\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{\cos x}{x} = 0$

(where x is measured in radians)

- **Limits of logarithmic and exponential functions**

(i) $\lim_{x\to\infty} a^x = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \\ \text{does not exist}, & a < 0 \end{cases}$

(ii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a\ (a > 0),\ \lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(iii) $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1,\ \lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$

(iv) If $\lim_{x\to a} f(x)$ exists and positive then

$\lim_{x\to a} [f(x)]^{\varphi(x)} = e^{\lim_{x\to a} \varphi(x).\ln[f(x)]}$

(v) If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} \varphi(x) = \infty$

then $\lim_{x\to a} f(x)^{\phi(x)} = \lim_{x\to a} [1 + f(x) - 1)]^{\phi(x)}$

$= e^{\lim_{x\to a} (f(x)-1).\phi(x)}$

(vi) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to 0} (1+x)^{1/x} = e,$

$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x\to 0} (1+x)^{a/x} = e^a$

- **Some Standard Methods to Find Limit of a Function**
  - Factorization
  - Rationalization
  - By application of standard limits

- Expansion of function
- Substitution
- By use of logarithm

L' Hospital's Rule

Suppose that either $\lim_{x\to a} f(x)=0, \lim_{x\to a} g(x)=0$ or both the limits are ∞ or $-\infty$ and that f and g are differentiable on an open interval I containing a. Suppose also that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

CONTINUITY AND DISCONTINUITY OF A FUNCTION AT A POINT

A function $f(x)$ is said to be continuous at $x = a$, if $\lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=f(a)$

i.e. L.H.L. = R.H.L. = value of the function at 'a'

i.e. $\lim_{x\to a} f(x)=f(a)$.

If $f(x)$ is not continuous at $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

Note :

(i) All polynomials, trigonometrical functions, exponential and logarithmic functions are continuous in their domain.

(ii) We never talk about continuity/discontinuity at point at which we can't approach from either side of the point. These points are called isolated points *e.g.* $f(x)=\sqrt{a-x}+\sqrt{x-a}$ at $x=a$

(iii) There are some functions which are continuous only at one point.

$$e.g.\ f(x)=\begin{cases} x & \text{if } x\in Q \\ -x & \text{if } x\notin Q \end{cases}$$

$$\text{and } g(x)=\begin{cases} x & \text{if } x\in Q \\ 0 & \text{if } x\notin Q \end{cases}$$

are both continuous only at $x = 0$

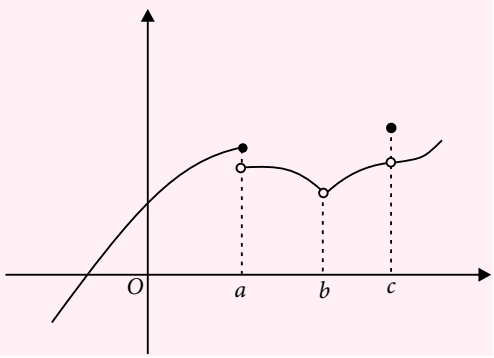
Reasons of Discontinuity

(i) $\lim_{x\to a} f(x)$ does not exist.

(ii) $f(x)$ is not defined at $x = b$.

(iii) $\lim_{x\to c} f(x)\neq f(c)$.

In all the above cases, geometrically the graph of the function will exhibit a break at the point of discussion. The graph as shown is discontinuous at $x = a$, b and c.



CONTINUITY ON AN INTERVAL

- **Continuity on an Open Interval**

  A function $f(x)$ is said to be continuous in an open interval (a, b) if it is continuous at each and every point of (a, b)

- **Continuity on a Closed Interval**

  A function $f(x)$ is said to be continuous in a closed interval $[a, b]$ if

  (a) It is continuous in (a, b)

  (b) Value of the function at "b" is equal to left hand limit at "b" *i.e.*, $f(b) = \lim_{x\to b^-} f(x)$

  (c) Value of the function at "a" is equal to right hand limit at "a" *i.e.*, $f(a) = \lim_{x\to a^+} f(x)$

- **Algebra of Continuous Functions**

  Let $f(x)$ and $g(x)$ are continuous functions at $x = a$. Then,

  (i) $cf(x)$ is continuous at $x = a$, where c is any constant

  (ii) $f(x) \pm g(x)$ is continuous at $x = a$

  (iii) $f(x)\cdot g(x)$ is continuous at $x = a$

  (iv) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$

  Note :

  (a) If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x) = f(x)\cdot g(x)$ is not necessarily be discontinuous at $x = a$.

  (b) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x)\cdot g(x)$ is not necessarily be discontinuous at $x = a$.

  (c) Continuity of an inverse function : If the function $y = f(x)$ is defined, continuous and strictly monotonic on the interval X, then there exist a single valued inverse function $x = \varphi(y)$ defined, continuous and also strictly monotonic in the range of the function $y = f(x)$.

- **Continuity of Composite Function**

If f is continuous at $x = c$ and g is continuous at $x = f(c)$ then the composite $g(f(x))$ is continuous at $x = c$.

APPLICATIONS OF CONTINUITY

- **Intermediate value theorem**

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, there exists a number c between a and b such that $f(c) = y_0$.

- **Extreme value theorem**

If f is continuous at every point of a closed interval I, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere in I. That is, there are numbers x_1 and x_2 in I with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in I.

DEFINITION OF THE DERIVATIVE

The derivative $f'(x)$ of a function $y = f(x)$ at a given point x is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \text{finite}.$$

If this limit exists finitely then the function $f(x)$ is called differentiable at the point x. The number $f'_-(x)\left(= \lim_{\Delta x \to 0^-} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \text{finite} \right)$ is called the left hand derivative at the point x.

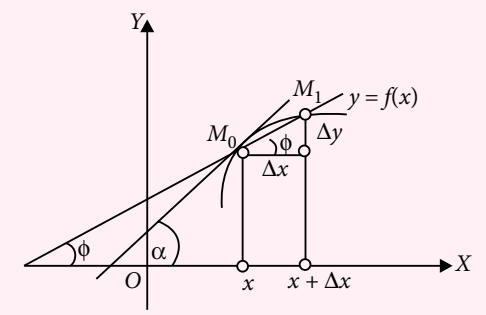
Similarly the number

$f'_+(x)\left(= \lim_{\Delta x \to 0^+} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \text{finite} \right)$

is called the right hand derivative at the point x. The necessary and sufficient condition for the existence of the derivative $f(x)$ is the existence of the finite right and left hand derivatives, and also of the equality $f'_+(x) = f'_-(x) =$ finite.

- **Geometrical Meaning of the Derivative**

Let us consider the function $f(x)$ and the corresponding curve $y = f(x)$. Clearly line joining two points M_0 (x, y) and M_1 $(x + \Delta x, y + \Delta y)$ on the curve will be the secant to the curve and the slope of this secant is given by $\tan\phi = \frac{\Delta y}{\Delta x}$ (Where ϕ is the angle made by the secant with the positive direction of the x-axis). In the limiting case when $\Delta x \to 0$ the point M_1 approaches M_0 and the secant joining these two points will become the tangent at M_0 whose slope will be given by $\tan\alpha = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$ which means that slope of the tangent to the curve $y = f(x)$ at any argument is equal to the value of the derivative at that argument.



- **Physical meaning of Derivative**

The (instantaneous) velocity is the derivative of the position function $s = f(t)$ with respect to time. At time t, the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

DIFFERENTIABILITY ON AN INTERVAL

A function $y = f(x)$ is differentiable on an interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it differentiable at every point of the open interval (a, b) and if the limits $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$ = finite (Right - hand derivative at a)

$\lim_{h \to 0^+} \frac{f(b) - f(b-h)}{h}$ = finite (Left - hand derivative at b) exist finitely

- **Causes of Non-Derivability**

(i) a corner, where function is continuous but L.H.D. and R.H.D. differ and are finite.

(ii) a cusp, where the slope approaches ∞ from one side and $-\infty$ from the other.

(iii) a vertical tangent where the slope approaches ∞ from both sides or approaches $-\infty$ from both sides.

(iv) a discontinuity

- **Relation Between Derivability And Continuity**

(a) If $f'(a)$ exists then $f(x)$ is derivable at $x = a$ $\Rightarrow f(x)$ is continuous at $x = a$. In general a function f is derivable at x then f is continuous at x. *i.e.* if $f(x)$ is derivable for every point of its domain of definition, then it is continuous

in that domain. The converse of the above result is need not be true *e.g.* the functions $f(x) = |x|$ and $g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0, & x = 0 \end{cases}$ both are continuous at $x = 0$ but not derivable at $x = 0$.

(b) Let $f'_+(a) = \lambda$ and $f'_-(a) = \mu$, where λ and μ are finite then :

(i) $\lambda = \mu \Rightarrow f$ is derivable at $x = a$
$\Rightarrow$ f is continuous at $x = a$.

(ii) $\lambda \neq \mu \Rightarrow f$ is not derivable at $x = a$ but f is continuous at $x = a$. If a function f is not differentiable but is continuous at $x = a$, it geometrically implies a sharp corner or kink at $x = a$.

(iii) If f is not continuous at $x = a$ then it is not differentiable at $x = a$.

- **Algebra of Derivatives**

If u and v are derivable functions of x, then

(i) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(ii) $\frac{d}{dx}(Ku) = K\frac{du}{dx}$, where K is any constant

(iii) $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$
known as " Product Rule "

(iv) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$
where $v \neq 0$ known as " Quotient Rule "

(v) If $y = f(u)$ and $u = g(x)$ then
$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ known as "Chain Rule"

- **Derivative of Composite Function**

If $f(u)$ is differentiable at the point $u = g(x)$, and $g(x)$ is differentiable at x, then the composite function $(fog)(x) = f(g(x))$ is differentiable at x, and $(fog)'(x) = f'(g(x)).g'(x)$.

- **Derivative of $f(x)$ From The First Principle/ab Initio Method**

If $f(x)$ is a derivable function, then

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) = \frac{dy}{dx}$$

or simply $f'(x) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h}$

- **Some Standard Formulae of Differentiation**

(i) $\frac{d}{dx}(\text{constant}) = 0$ (ii) $\frac{d}{dx}x^n = nx^{n-1}$

(iii) $\frac{d}{dx}a^x = a^x \ln a$ (iv) $\frac{d}{dx}e^x = e^x$

(v) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}\ln|x| = \frac{1}{x}$ (vii) $\frac{d}{dx}\sin x = \cos x$

(viii) $\frac{d}{dx}\cos x = -\sin x$ (ix) $\frac{d}{dx}\tan x = \sec^2 x$

(x) $\frac{d}{dx}\sec x = \sec x \tan x$

(xi) $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

(xii) $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

- **Derivative of Inverse Trigonometric Functions**

(i) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(ii) $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

(iii) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, x \in R$

(iv) $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}, x \in R$

(iv) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

(vi) $\frac{d}{dx}\operatorname{cosec}^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

- **Differentiation of Implicit Function**

If the relation between the variables x and y is given by an equation containing both, and this equation is not immediately solvable for y, then y is called an implicit function of x. Implicit functions are given by $\phi(x, y) = 0$.

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a function of x and then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit functions, both x and y are present.

- **Logarithmic Differentiation : To find the derivative of :**

  (i) a function which is the product or quotient of a number of functions.

  OR

  (ii) a function of the form $[f(x)]^{g(x)}$, where f and g are both derivable, it will be found convenient to take the logarithm of the function first and then differentiate. This is called Logarithmic Differentiation.

- **Differentiation of A Function Defined Parametrically**

  Let x and y be the functions of parameter t, *i.e.*, $x = f(t)$, $y = \phi(t)$, then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\varphi'(t)}{f'(t)}$

- **Differentiation of A Function With Respect To Another Function**

$$\frac{df(x)}{d\varphi(x)} = \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}\varphi(x)} = \frac{f'(x)}{\varphi'(x)}$$

- **Second and Higher Order Derivatives**

The derivative $y' = \dfrac{dy}{dx}$ is the first (first order) derivative of y with respect to x. This derivative may itself be a differentiable function of x; if so, its derivative $y'' = \dfrac{dy'}{dx} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ is called the second (second order) derivative of y with respect to x.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the third (third order) derivative of y with respect to x. In the same way, $y^{(n)} = \dfrac{d}{dx}y^{(n-1)}$ denoting the n^{th} (n^{th} order) derivative of y with respect to x, for any positive integer n.

- **Relation Between $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^2x}{dy^2}$:**

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right) \div \frac{dy}{dx}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) \div \frac{dy}{dx}$$

$$= -\frac{d^2y}{dx^2} \div \left(\frac{dy}{dx}\right)^3 = -\frac{(y'')}{(y')^3}$$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. If a_n and b_n are positive integers and $a_n + \sqrt{2}\, b_n = (2+\sqrt{2})^n$, then $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right) =$

(a) $\sqrt{2}$ (b) 2 (c) $e^{\sqrt{2}}$ (d) e^2

2. If $f(0) = 0$ and that 'f is differentiable at $x = 0$, and 'k' is a positive integer. Then

$$\lim_{x\to 0}\frac{1}{x}\left[f(x) + f\left(\frac{x}{2}\right) + f\left(\frac{x}{3}\right) + \ldots\ldots + f\left(\frac{x}{k}\right)\right] =$$

(a) $k.f'(0)$ (b) $\left(\sum\limits_{r=1}^{k}\frac{1}{r}.\right)f'(0)$

(c) $\sum\limits_{r=1}^{k}\frac{1}{r}$ (d) does not exist

3. $\lim\limits_{x\to 0}\left(\sum\limits_{r=1}^{n} r^{\operatorname{cosec}^2 x}\right)^{\sin^2 x} =$

(a) 0 (b) ∞ (c) n (d) $\dfrac{1}{n}$

4. For each positive integer n, let

$$S_n = \frac{3}{1\cdot 2\cdot 4} + \frac{4}{2\cdot 3\cdot 5} + \frac{5}{3\cdot 4\cdot 6} + \ldots\ldots + \frac{n+2}{n(n+1)(n+3)}$$

Then $\lim\limits_{n\to\infty} S_n$ equals

(a) $\dfrac{29}{6}$ (b) $\dfrac{29}{36}$ (c) 0 (d) $\dfrac{29}{18}$

5. $\lim\limits_{x\to 0}\dfrac{a^{\tan x} - a^{\sin x}}{\tan x - \sin x}$ (where $a > 0$) is equal to

(a) $\log_e a$ (b) 1 (c) 0 (d) ∞

6. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{(1-\sin x)(8x^3 - \pi^3)\cos x}{(\pi - 2x)^4} =$

(a) $-\dfrac{\pi^2}{16}$ (b) $\dfrac{3\pi^2}{16}$ (c) $\dfrac{\pi^2}{16}$ (d) $-\dfrac{3\pi^2}{16}$

7. Let $f(x) = x^3\left\{\sqrt{x^2 + \sqrt{x^4+1}} - x\sqrt{2}\right\}$, then $\lim\limits_{x\to\infty} f(x)$ is equal to

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{4\sqrt{2}}$

(c) $\frac{3}{4\sqrt{2}}$ (d) does not exist

8. $\lim_{x\to\left(\frac{-1}{3}\right)^-} \frac{1}{x}\left[\frac{-1}{x}\right]$, [.] denotes greatest integer function is equal to

(a) –9 (b) –12 (c) –6 (d) 0

9. $\lim_{n\to\infty} \sum_{r=1}^{n} \cot^{-1}\left(r^2+\frac{3}{4}\right) =$

(a) $\tan^{-1}(2)$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) $\tan^{-1}(3)$

10. A function $f(x)$ is defined by,

$$f(x)=\begin{cases}\frac{[x^2]-1}{x^2-1}, \text{for } x^2\neq 1\\ 0 \qquad , \text{for } x^2=1\end{cases}$$ where [.] denotes GIF,

then $f(x)$ is

(a) Continuous at $x = -1$

(b) Discontinuous at $x = 1$

(c) Differentiable at $x = 1$

(d) None of these

11. Let 'f' be a real valued function defined on the interval (–1, 1) such that $e^{-x}\cdot f(x)=2+\int_0^x \sqrt{t^4+1}\,dt$ $\forall\, x\in(-1,1)$ and let 'g' be the inverse function of 'f'. Then $g'(2) =$

(a) 3 (b) 1/2 (c) 1/3 (d) 2

12. $f(x) = \max(3 - x, 3 + x, 6)$ is differentiable at

(a) All points

(b) No point

(c) All points except two points

(d) All points except at one point

13. The derivative of $f(\tan x)$ with respect to $g(\sec x)$ at $x=\frac{\pi}{4}$, if $f'(1)=2, g'(\sqrt{2})=4$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) 1

14. The value of $\lim_{x\to\infty}\left(\frac{a_1^{1/x}+a_2^{1/x}+....+a_n^{1/x}}{n}\right)^{nx}$ is

(a) $a_1 + a_2 + ... a_n$ (b) $e^{a_1+a_2+.....+a_n}$

(c) $\frac{a_1+a_2+.....+a_n}{n}$ (d) $a_1a_2 .....a_n$

15. The number of points at which the function $f(x) = \max.\{a - x, a + x, b\}$, $-\infty < x < \infty$, $0 < a < b$ can not be differentiable is

(a) 2 (b) 3 (c) 1 (d) 0

SECTION-II
Multiple Correct Answer Type

16. If $f(x)=x\left(\frac{e^{|x|+[x]}-2}{|x|+[x]}\right)$ then

(a) $\lim_{x\to0^+} f(x)=-1$ (b) $\lim_{x\to0^-} f(x)=0$

(c) $\lim_{x\to0} f(x)=-1$ (d) $\lim_{x\to0} f(x)=0$

17. Suppose 'f' is a function that satisfies the equation $f(x + y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers 'x' and 'y'. If $\lim_{x\to0}\frac{f(x)}{x}=1$, then

(a) $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$

(b) $f'(0) = 1$

(c) $f''(0) = 1$

(d) $f'''(0) = 6$

18. If $f(x)=\lim_{p\to\infty}\frac{x^p g(x)+h(x)+7}{7x^p+3x+1}; x\neq1$ and $f(1) = 7$, $f(x)$, $g(x)$ and $h(x)$ are all continuous functions at $x = 1$. Then which of the following statement(s) is/are correct?

(a) $g(1) + h(1) = 70$ (b) $g(1) - h(1) = 28$

(c) $g(1) + h(1) = 60$ (d) $g(1) - h(1) = -28$

19. The function $f(x)=\frac{1}{u^2-u-2}$ where $u=\frac{1}{x-1}$

(a) has a removable discontinuity at $x = 1$

(b) has irremovable discontinuity at $x = 0$, $\frac{3}{2}$

(c) is discontinuous at $u = -2, 1$

(d) is discontinuous at $u = -1, 2$

20. If $\lim_{x\to0}\frac{a\sin x-bx+cx^2+x^3}{2x^2\ln(1+x)-2x^3+x^4}$ exists and is finite, then

(a) $a = 6$ (b) $b = 0$

(c) $c = 0$ (d) The limit = $\frac{3}{40}$

21. If $\lim_{x\to0}\sin\left(\frac{\pi\left(1-\cos^m x\right)}{x^n}\right)$ exists, where $m, n \in N$, then

(a) $m \in N, n = 3$ (b) $m \in N, n \in N$

(c) $m \in N, n = 2$ (d) $m \in N, n = 1$

22. α, β are roots of equation $ax^2 + bx + c = 0$ where $1 < \alpha < \beta$. If $\lim_{x\to m} \frac{|ax^2+bx+c|}{ax^2+bx+c} = 1$ then which of the following are true?

(a) $a < 0$ and $\alpha < m < \beta$
(b) $a > 0$ and $m < 1$
(c) $a > 0$ and $\alpha < m < \beta$
(d) $a > 0$ and $m > 1$

23. A function $f : R \to R$ satisfies the equation $f(x + y) = f(x) \cdot f(y)$ for all x, y in R and $f(x) \neq 0$ for any $x \in R$. Let the function be differentiable at $x = 0$ and $f'(0) = 2$ then

(a) $f'(x) = 2f(x)\ \forall\ x \in R$
(b) $f(x) = e^{2x}$
(c) $f(x)$ is everywhere continuous
(d) $f\left(\frac{1}{2}\right)$ is an irrational number

24. If $8f(x) + 6f\left(\frac{1}{x}\right) = x + 5$ where $x \neq 0$ and $y = x^2 f(x)$ then

(a) $f(x) = \frac{1}{28}\left[8x - \frac{6}{x} + 10\right]$
(b) $\left(\frac{dy}{dx}\right) = \frac{1}{28}[24x^2 - 6 + 20x]$
(c) $\left(\frac{dy}{dx}\right)_{(x=-1)} = -\frac{1}{14}$
(d) $\left(\frac{dy}{dx}\right)_{(x=1)} = \frac{19}{14}$

25. Suppose that $f: R \to R$ is continuous and satisfy the equation $f(x) \cdot f(f(x)) = 1$, for all real x.
Let $f(1000) = 999$, then which of the following is true ?

(a) $f(500) = \frac{1}{500}$ (b) $f(199) = \frac{1}{199}$
(c) $f(x) = \frac{1}{x} \forall\ x \in R - \{0\}$ (d) $f(1999) = \frac{1}{1999}$

SECTION-III
Comprehension Type

Paragraph for Question No. 26 to 28

Let $n \in N$, The A.M, G.M, H.M of the n numbers $n + 1, n + 2, n + 3, \ldots, n + n$ are A_n, G_n, H_n respectively then

26. $\lim_{n\to\infty} \frac{A_n}{n} =$
(a) 1 (b) 1/2 (c) 3/2 (d) 2

27. $\lim_{n\to\infty} \frac{G_n}{n} =$
(a) $1/e$ (b) $2/e$ (c) $3/e$ (d) $4/e$

28. $\lim_{n\to\infty} \frac{H_n}{n} =$
(a) $1/e$ (b) 1/log2
(c) $2/e$ (d) 1/log4

Paragraph for Question No. 29 to 31

If f, g and h are functions having a common domain D and $h(x) \le f(x) \le g(x), x \in D$ and if $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = l$ then $\lim_{x\to a} f(x) = l$. This is known as Sandwich Theorem.
Using this result, compute the following limits.

29. The value of $\lim_{x\to 0} \frac{|x|}{\sqrt{x^4 + 4x^2 + 7}}$ is
(a) 1 (b) 0
(c) 1/2 (d) does not exist

30. $\lim_{x\to 0} x^4 \sin\left(\frac{1}{3\sqrt{x}}\right)$ is
(a) 0 (b) 1
(c) 1/3 (d) does not exist

31. Let $f(x) = x^2\left(\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), x \neq 0$, then
(a) $\lim_{x\to 0^+} f(x)$ doesn't exist
(b) $\lim_{x\to 0} f(x)$ doesn't exist
(c) $\lim_{x\to 0} f(x)$ exist (d) $\lim_{x\to 0} f(x) = 1$

Paragraph for Question No. 32 to 34

In the evaluation of limits following the paragraph, one may use one or the other of the following results:
(1) If $f(x)$ and $g(x)$ are functions defined in some deleted neighbourhood N of 'a' such that $g(x)$ never vanishes in N and $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a} g(x) = 0$ implies $\lim_{x\to a} f(x) = 0$
(2) If $f(x) \to 1$ as $x \to a$, $g(x) \to +\infty$ as $x \to a$ and $\lim_{x\to a} \{f(x)\}^{g(x)}$ exists, then this limit equals $e^{\lim_{x\to a} g(x)\{f(x)-1\}}$

32. If $\lim_{x\to 0} \frac{\sin ax - \log_e(e^x \cos x)}{x \sin bx} = \frac{1}{2}$, then
(a) $a = 1, b = 1$ (b) $a = -1, b = \frac{1}{2}$
(c) $a = -1, b = 1$ (d) $a = -1, b = \frac{1}{2}$

33. Let $f(x)$ be a function that is defined in a deleted neighbourhood of '0' such that $\lim_{x\to 0}\frac{f(x)}{x}=1$. If $\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{\{f(x)\}^3}=1$ then

(a) $a=-\frac{3}{2}, b=-\frac{1}{2}$ (b) $a=\frac{1}{2}, b=\frac{3}{2}$

(c) $a=-\frac{5}{2}, b=-\frac{3}{2}$ (d) $a=\frac{3}{2}, b=\frac{5}{2}$

34. Let $L=\lim_{x\to 0}\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{2}}{x^2\sin^2 x}$, $a>0$. If L is finite, then $L=$

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

SECTION-IV
Matrix-Match Type

35. Let $f(x)=\lim_{n\to\infty}\frac{x^{2n}-1}{x^{2n}+1}$

Column-I		Column-II	
(A)	$f=1$ on	(p)	$(1, \infty)$
(B)	$f=-1$ on	(q)	$(2, \infty)$
(C)	$f\geq 0$ on	(r)	$[2, \infty)$
(D)	$f(x)=\text{sgn}\,(\lvert x\rvert-1)$	(s)	$\left(-\frac{1}{2}, \frac{1}{2}\right)$

36. Match the following.

Column-I		Column-II	
(A)	$f(x)=x\,\text{sgn}\,(x-1)$	(p)	$\lim_{x\to 1} f(x)$ doesn't exist
(B)	$f(x)=\frac{\sin(\sin(\tan(x^2/2)))}{\log\cos 3x}$	(q)	$\lim_{x\to 0} f(x)$ doesn't exist
(C)	$f(x)=\frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$	(r)	$\lim_{x\to 0} f(x)=-1/9$
(D)	$f(x)=\frac{e^{1/x}-1}{e^{1/x}+1}$	(s)	$\lim_{x\to 0} f(x)=-1$

37. Match the following.

Column-I		Column-II	
(A)	$\lim_{x\to 3}\frac{(x^3+27)\log(x-2)}{x^2-9}=$	(p)	12
(B)	$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\left(\frac{x}{x+1-e^x}\right)}=$	(q)	8
(C)	If $\lim_{x\to 0}\frac{x(a+\cos x)-b\sin x}{x^3}=1$ then a and b are respectively	(r)	9
(D)	If $f(x)$ is a thrice differentiable function such that $\lim_{x\to 0}\frac{f(4x)-3f(3x)+3f(2x)-f(x)}{x^3}=12$ then $f'''(0)$ is equal to	(s)	e^{-1}

SECTION-V
Integer Answer Type

38. If $f(n+1)=\frac{1}{2}\left\{f(n)+\frac{9}{f(n)}\right\}$ where $n\in N$ and $f(n) > 0\ \forall\ n\in N$ and $\lim_{n\to\infty} f(n)$ exist then the value of $\lim_{n\to\infty} f(n)=$

39. The integer 'n' for which $\lim_{x\to 0}\left[\frac{(\cos x-1)(\cos x-e^x)}{x^n}\right]$ is a finite non-zero number, is

40. If $\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2$ then the value of $a+b+c=$

41. The value of $\lim_{x\to\frac{\pi}{2}}\sqrt{\frac{\tan x-\sin(\tan^{-1}(\tan x))}{\tan x+\cos^2(\tan x)}}$ is

42. Let $(\tan\alpha)x + (\sin\alpha)y = \alpha$ and $(\alpha\text{cosec}\,\alpha)x + (\cos\alpha)y = 1$ be two variable straight lines, α being the parameter. Let P be the point of intersection of the lines. If the coordinates of P in the limiting position when $\alpha\to 0$ be (h, k), then $h-k$ is equal to

43. The function $f(x) = |x^2 - 3x + 2| + \cos|x|$ is not differentiable at how many values of x.

44. If the function f defined by $f(x)=\dfrac{x(1+a\cos x)-b\sin x}{x^3}$, $x \neq 0$ and $f(0) = 1$ is continuous at $x = 0$, then $2a - 8b =$

45. If $f\left(\dfrac{x+y}{2}\right)=\dfrac{f(x)+f(y)}{2}$ for all $x, y \in R$, $f'(0)$ exists and equals to -1 and $f(0) = 1$ then $5 - f(2) =$

SOLUTIONS

1. (a) : We have, $a_n+\sqrt{2}\,b_n=(2+\sqrt{2})^n$

$\Rightarrow a_n-\sqrt{2}\,b_n=(2-\sqrt{2})^n$

Therefore $a_n=\dfrac{1}{2}\left[(2+\sqrt{2})^n+(2-\sqrt{2})^n\right]$

and $b_n=\dfrac{\left[(2+\sqrt{2})^n-(2-\sqrt{2})^n\right]}{2\sqrt{2}}$

Therefore $\dfrac{a_n}{b_n}=\sqrt{2}\dfrac{\left[(2+\sqrt{2})^n+(2-\sqrt{2})^n\right]}{\left[(2+\sqrt{2})^n-(2-\sqrt{2})^n\right]}$

$$=\sqrt{2}\frac{\left[1+\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\dfrac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence $\lim\limits_{n\to\infty}\left(\dfrac{a_n}{b_n}\right)=\sqrt{2}\left(\dfrac{1+0}{1-0}\right)\quad\left(\because\dfrac{2-\sqrt{2}}{2+\sqrt{2}}<1\right)$

$=\sqrt{2}$

2. (b) : $l=\lim\limits_{x\to0}\left\{\dfrac{f(x)-f(0)}{x-0}+\dfrac{f\left(\dfrac{x}{2}\right)-f(0)}{x-0}+....\right.$

$\left.....+\dfrac{f\left(\dfrac{x}{k}\right)-f(0)}{x-0}\right\}$

$=\left(1+\dfrac{1}{2}+\dfrac{1}{3}+....+\dfrac{1}{k}\right)f'(0).$

3. (c) : $L=\lim\limits_{x\to0}(1^{\text{cosec}^2x}+2^{\text{cosec}^2x}+....+n^{\text{cosec}^2x})^{\sin^2x}$

$$=\lim_{x\to0}\left(\left(\frac{1}{n}\right)^{\text{cosec}^2x}+\left(\frac{2}{n}\right)^{\text{cosec}^2x}+....\right.$$
$$\left.....+\left(\frac{n-1}{n}\right)^{\text{cosec}^2x}+1\right)^{\sin^2x}\cdot n$$

$= (0 + 0 + 0 +....+ 1)^0 \cdot n = n$

4. (b) : Let $u_k=\dfrac{k+2}{k(k+1)(k+3)}$

$=\dfrac{(k+2)^2}{k(k+1)(k+2)(k+3)}=\dfrac{k^2+4k+4}{k(k+1)(k+2)(k+3)}$

$=\dfrac{k(k+1)+3k+4}{k(k+1)(k+2)(k+3)}$

$=\dfrac{1}{(k+2)(k+3)}+\dfrac{3}{(k+1)(k+2)(k+3)}$

$+\dfrac{4}{k(k+1)(k+2)(k+3)}$

$=\left(\dfrac{1}{k+2}-\dfrac{1}{k+3}\right)-\dfrac{3}{2}\left[\dfrac{1}{(k+2)(k+3)}\right.$

$\left.-\dfrac{1}{(k+1)(k+2)}\right]$

$-\dfrac{4}{3}\left[\dfrac{1}{(k+1)(k+2)(k+3)}-\dfrac{1}{k(k+1)(k+2)}\right]$

Now, put $k = 1, 2, 3, ..., n$ and add. Thus

$S_u = u_1 + u_2 + ... + u_n$

$=\left(\dfrac{1}{3}-\dfrac{1}{n+3}\right)-\dfrac{3}{2}\left[\dfrac{1}{(n+2)(n+3)}-\dfrac{1}{2.3}\right]$

$-\dfrac{4}{3}\left[\dfrac{1}{(n+1)(n+2)(n+3)}-\dfrac{1}{1.2.3}\right]$

Therefore $\lim\limits_{n\to\infty}S_n=\dfrac{1}{3}+\dfrac{3}{12}+\dfrac{4}{18}=\dfrac{29}{36}$

5. (a) : We have

$\lim\limits_{x\to0}\dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}=\lim\limits_{x\to0}a^{\sin x}\left(\dfrac{a^{\tan x-\sin x}-1}{\tan x-\sin x}\right)$

$=\lim\limits_{x\to0}(a^{\sin x})\times\lim\limits_{t\to0}\left(\dfrac{a^t-1}{t}\right)$ (where $t=\tan x-\sin x$)

$= a^0 \times \log_e a = \log_e a$

6. (d) : Let $f(x)=\dfrac{(1-\sin x)(8x^3-\pi^3)\cos x}{(\pi-2x)^4}$

$=\dfrac{(1-\sin x)\cos x(2x-\pi)(4x^2+2\pi x+\pi^2)}{(2x-\pi)^4}$

$=\dfrac{(1-\sin x)\cos x(4x^2+2\pi x+\pi^2)}{(2x-\pi)^3}$

Therefore, $\lim\limits_{x\to\frac{\pi}{2}}f(x)=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{(1-\sin x)\cos x}{(2x-\pi)^3}.(3\pi^2)$ ...(i)

Put $2x - \pi = y$ so that $y \to 0$ as $x \to \pi/2$. Therefore now

$$\frac{(1-\sin x)\cos x}{(2x-\pi)^3}=\frac{\left[1-\sin\left(\frac{\pi+y}{2}\right)\right]\cos\left(\frac{\pi+y}{2}\right)}{y^3}$$

$$=\frac{\left(1-\cos\frac{y}{2}\right)\left(-\sin\frac{y}{2}\right)}{y^3}=-\left(\frac{2\sin^2\frac{y}{4}}{y^2}\right)\left(\frac{\sin\frac{y}{2}}{y}\right)$$

$$=-2\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2\cdot\frac{1}{16}\cdot\left(\frac{\sin\frac{y}{2}}{y/2}\right)\cdot\frac{1}{2}=\frac{-1}{16}\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2\left(\frac{\sin\frac{y}{2}}{y/2}\right)\quad\text{...(ii)}$$

Therefore from (i) and (ii), we get

$$\lim_{x\to\frac{\pi}{2}}f(x)=\frac{-3\pi^2}{16}\times1\times1$$

7. (b) : We have, $f(x)=\dfrac{x^3\{x^2+\sqrt{x^4+1}-2x^2\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$

$$=\frac{x^3\{\sqrt{x^4+1}-x^2\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$$

$$=\frac{x^3(x^4+1-x^4)}{\left[\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$=\frac{x^3}{\left[\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$=\frac{1}{\left[\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\right]\left[\sqrt{1+\frac{1}{x^4}}+1\right]}$$

$$\therefore\quad\lim_{x\to\infty}f(x)=\frac{1}{(\sqrt{1+\sqrt{1}}+\sqrt{2})(\sqrt{1}+1)}=\frac{1}{2\sqrt{2}(2)}=\frac{1}{4\sqrt{2}}$$

8. (c) : Now, $x<-\frac{1}{3},\ \frac{1}{x}>-3\ \Rightarrow\ -\frac{1}{x}<3\ \Rightarrow\left[-\frac{1}{x}\right]=2$

Now, $\lim\limits_{x\to\left(-\frac{1}{3}\right)^-}\frac{1}{x}\left[-\frac{1}{x}\right]=(-3)(2)=-6$

9. (a) : $\cot^{-1}\left(r^2+\frac{3}{4}\right)=\tan^{-1}\left(\frac{1}{r^2+\frac{3}{4}}\right)$

$$=\tan^{-1}\left(\frac{1}{1+\left(r^2-\frac{1}{4}\right)}\right)$$

$$=\tan^{-1}\left(\frac{1}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)$$

$$=\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)$$

$$=\tan^{-1}\left(r+\frac{1}{2}\right)-\tan^{-1}\left(r-\frac{1}{2}\right)$$

10. (b) : $f(x)=\begin{cases}\dfrac{[x^2]-1}{x^2-1}, & \text{for } x^2\neq1\\ 0, & \text{for } x^2=1\end{cases}$

$$=\begin{cases}\dfrac{-1}{x^2-1}, & \text{for } 0<x^2<1\\ 0, & \text{for } x^2=1\\ 0, & \text{for } 1<x^2<2\end{cases}$$

$\therefore$ R.H.L at $x=1$ is 0

Also L.H.L. at $x=1$ is $-\infty$

11. (c) : Differentiating given equation, we get

$$e^{-x}.f'(x)-e^{-x}\cdot f(x)=\sqrt{1+x^4}\qquad\text{...(i)}$$

Since $(gof)(x)=x$ as 'g' is inverse of f.

$\Rightarrow\ g[f(x)]=x\ \Rightarrow\ g'[f(x)].f'(x)=1$

$\Rightarrow\ g'[f(0)]=\dfrac{1}{f'(0)}\ \Rightarrow\ g'(2)=\dfrac{1}{f'(0)}$

(Here $f(0)=2$ observe from hypothesis)

Put $x=0$ in (i), we get $f'(0)=3$

12. (c) : $f(x)=\begin{cases}3-x, & x<-3\\ 6, & -3\le x\le3\\ 3+x, & x>3\end{cases}$

Since these expressions are linear functions in x or a constant

It is clearly differentiable at all points except at the border points at –3 and 3

At $x=-3$, L.H.D. = –1, R.H.D. = 0

At $x=3$, L.H.D. = 0, R.H.D. = 1

At $x=-3$ and $x=3$, it is not differentiable.

13. (a) : Let $u=f(\tan x)$

$$\frac{du}{dx}=f'(\tan x)\sec^2x$$

$v=g(\sec x)$

$$\frac{dv}{dx} = g'(\sec x)\sec x\tan x$$

Now $\left(\frac{du}{dv}\right)_{x=\frac{\pi}{4}} = \frac{f'(\tan x)\sec^2 x}{g'(\sec x)\sec x\tan x} = \frac{f'(1)2}{g'(\sqrt{2})\sqrt{2}}$

$$= \frac{2\cdot 2}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

14. (d) : Let $x = \frac{1}{y}$ then, $x \to \infty$, $y \to 0$

$$= \lim_{x\to\infty}\left(\frac{a_1^{1/x} + a_2^{1/x} + \ldots\ldots\ldots + a_n^{1/x}}{n}\right)^{nx}$$

$$= \lim_{y\to 0}\left(\frac{a_1^y + a_2^y + \ldots\ldots\ldots + a_n^y}{n}\right)^{n/y} \qquad (1^\infty \text{ form})$$

$$= \lim_{y\to 0}\left(1 + \frac{a_1^y + a_2^y + \ldots\ldots\ldots + a_n^y - n}{n}\right)^{n/y}$$

$$= \lim_{y\to 0}\frac{n}{y}\left(\frac{a_1^y + a_2^y + \ldots\ldots\ldots a_n^y}{n} - 1\right)$$

$$= e^{\lim_{y\to 0}\left(\frac{a_1^y - 1}{y} + \frac{a_2^y - 1}{y} + \ldots\ldots + \frac{a_n^y - 1}{y}\right)}$$

$$= e^{\log a_1 + \log a_2 + \log a_3 + \ldots\ldots + \log a_n}$$

$$= e^{\log(a_1a_2a_3\ldots\ldots\ldots\ldots a_n)} = (a_1a_2a_3\ldots\ldots\ldots a_n)$$

15. (a) : $f(x) = \begin{cases} a - x & \text{if} \quad x < a - b \\ b & \text{if} \quad a - b \le x \le b - a \\ a + x & \text{if} \quad x > b - a \end{cases}$

Hence f is not differentiable at $x = a - b$, $b - a$

16. (a, b) :

(a) $\lim_{x\to 0^+} x\left(\frac{e^{|x|+[x]} - 2}{|x| + [x]}\right) = \lim_{x\to 0^+} x\left(\frac{e^{x+0} - 2}{x + 0}\right)$

$$= \lim_{x\to 0^+}(e^x - 2) = 1 - 2 = -1$$

(b) $\lim_{x\to 0^-} x\left(\frac{e^{|x|+[x]} - 2}{|x| + [x]}\right) = \lim_{x\to 0^-} x\left(\frac{e^{-x-1} - 2}{-x - 1}\right) = 0$

17. (a, b) : Observe that $f(0) = 0$

and $f'(0) = \lim_{h\to 0}\frac{f(h)}{h} = 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h) + x^2h + xh^2}{h} = x^2 + 1$$

Hence, $f(x) = \frac{x^3}{3} + x$

18. (a, b) : When $x < 1$

$$f(1) = \frac{h(1) + 7}{3 + 1} \Rightarrow 7 = \frac{h(1) + 7}{4}$$

$\Rightarrow h(1) = 21$

When $x > 1$

$$f(1) = \lim_{p\to\infty}\frac{\left[g(x) + \frac{h(x)}{x^p} + \frac{7}{x^p}\right]}{\left[7 + \frac{3x+1}{x^P}\right]} \Rightarrow 7 = \frac{g(1)}{7}$$

$\therefore \quad g(1) = 49 \ \therefore \ g(1) - h(1) = 28$

and $g(1) + h(1) = 70$

19. (a, b, d) : The function $u = \frac{1}{x-1}$ is discontinuous at $x = 1$

$f(x) = \frac{1}{(u+1)(u-2)}$ is discontinuous at $u = -1, 2$

i.e., at $x = 0, \frac{3}{2}$

Also we have, $\lim_{x\to 0} f(x) = \lim_{u\to -1} f(x) = \infty$

$\lim_{x\to\frac{3}{2}} f(x) = \lim_{u\to 2} f(x) = \infty$

$\lim_{x\to 1} f(x) = \lim_{u\to\infty} f(x) = 0$

20. (a, c, d) : Given limit

$$= \lim_{x\to 0}\frac{a\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.\right) - bx + cx^2 + x^3}{2x^2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) - 2x^3 + x^4}$$

$$= \lim_{x\to 0}\frac{(a-b)x + cx^2 + \left(1 - \frac{a}{6}\right)x^3 + \frac{ax^5}{120}\ldots.}{2\frac{x^5}{3} - \frac{x^6}{2} + \ldots}$$

For this limit to exist, we must have

$a = b$, $c = 0$, $a = 6$

and given limit $= \frac{a}{120}\times\frac{3}{2} = \frac{6\times 3}{120\times 2} = \frac{3}{40}$

21. (c, d) : $\lim_{x\to 0}\sin\left(\frac{\pi\left(1 - \cos^m x\right)}{x^n}\right)$

$$= \sin\lim_{x\to 0}\left(\frac{\pi(1 - \cos^m x)}{x^n}\right)$$

Possible when $m \in N$ and $n = 1$ or 2.

22. (a, b) : $\lim_{x\to m}\frac{|a(x-\alpha)(x-\beta)|}{a(x-\alpha)(x-\beta)} = 1$

when $a > 0$, $m < 1 \Rightarrow (m - \alpha)(m - \beta) > 0$

$\Rightarrow\ a(m-\alpha)(m-\beta)>0$

$\Rightarrow \lim\limits_{x\to m}\dfrac{a(x-\alpha)(x-\beta)}{a(x-\alpha)(x-\beta)}=1$

when $a<0,\ \alpha<m<\beta \Rightarrow (m-\alpha)(m-\beta)<0$

$\Rightarrow\ a(m-\alpha)(m-\beta)>0$

$\Rightarrow \lim\limits_{x\to m}\dfrac{a(x-\alpha)(x-\beta)}{a(x-\alpha)(x-\beta)}=1$

23. (a, b, c, d) : Clearly for $x=y=0; f(0)=1$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=2\cdot f(x)$$

Integrating $f(x)=e^{2x}$, from this all the remaining follows.

24. (a, b, c, d) : $8f(x)+6f\left(\dfrac{1}{x}\right)=x+5$ ...(1)

Replacing x by $\dfrac{1}{x}$, we get

$$6f(x)+8f\left(\frac{1}{x}\right)=\frac{1}{x}+5 \qquad ...(2)$$

From (1) & (2)

$$f(x)=\frac{1}{28}\left(8x-\frac{6}{x}+10\right) \qquad ...(3)$$

$\because\quad y=x^2f(x)=\dfrac{1}{28}[8x^3-6x+10x^2]$

$$\frac{dy}{dx}=\frac{1}{28}[24x^2-6+20x]$$

$$\left(\frac{dy}{dx}\right)_{(x=-1)}=\frac{1}{28}[24-6-20]=-\frac{1}{14}$$

25. (a, b) : $f(1000)\,f(f(1000))=1$

$\Rightarrow\ f(1000)\,f(999)=1$

$\Rightarrow\ 999\,f(999)=1$

$\therefore\ f(999)=\dfrac{1}{999}$

The numbers 999 and $\dfrac{1}{999}$ are in the range of f. Hence by intermediate value property (IVP) of continuous function, function takes all values between 999 and $\dfrac{1}{999}$, then there exists

$\alpha\in\left(\dfrac{1}{999},999\right)$ such that $f(\alpha)=500$

Then $f(\alpha)\,f(f(\alpha))=1 \Rightarrow f(500)=\dfrac{1}{500}$

Similarly $199\in\left(\dfrac{1}{199},999\right)$, thus $f(199)=\dfrac{1}{199}$

But there is nothing to show that 1999 lies in the range of f.

Thus (d) is not correct and so (c) also.

26. (c) : $A_n=\dfrac{(n+1)+(n+2)+....+(n+n)}{n}$

$$=\frac{n^2+\frac{n(n+1)}{2}}{n}=\frac{3n+1}{2}$$

$$\frac{A_n}{n}=\frac{3}{2}+\frac{1}{2n}\to\frac{3}{2}\ \text{as}\ n\to\infty$$

27. (d) : $G_n=((n+1)(n+2)\ .....\ (n+n))^{1/n}$

$$=n\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right)^{1/n}$$

$$\frac{G_n}{n}=\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right)^{1/n}$$

$$\log\frac{G_n}{n}=\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\log\frac{G_n}{n}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$=\int_0^1\log(1+x)dx=\log\frac{4}{e}$$

$$\Rightarrow \lim_{n\to\infty}\frac{G_n}{n}=\frac{4}{e}$$

28. (b) : $H_n=\dfrac{n}{\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{n+n}}$

$$\Rightarrow\ \frac{n}{H_n}=\sum_{r=1}^{n}\frac{1}{n+r}\ \Rightarrow\ \lim_{n\to\infty}\frac{n}{H_n}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}=\int_0^1\frac{dx}{1+x}$$

$$=|\log(1+x)|_0^1=\log 2$$

29. (b) : Since $\sqrt{x^4+4x^2+7}\ge 1$, so

$$0\le\frac{|x|}{\sqrt{x^4+4x^2+7}}\le|x|,\ \text{But}\ \lim_{x\to 0}|x|=0$$

$$\text{Hence}\ 0\le\lim_{x\to 0}\frac{|x|}{\sqrt{x^4+4x^2+7}}\le\lim_{x\to 0}|x|=0$$

$$\therefore\ \lim_{x\to 0}\frac{|x|}{\sqrt{x^4+4x^2+7}}=0$$

30. (a) : Since $-1\le\sin\left(\dfrac{1}{3\sqrt{x}}\right)\le 1$,

so $-x^4\le x^4\sin\left(\dfrac{1}{3\sqrt{x}}\right)\le x^4$

But $\lim_{x\to 0} x^4 = 0$

Hence, $0 \le \lim_{x\to 0} x^4 \sin\left(\frac{1}{3\sqrt{x}}\right) \le 0$

$\therefore \quad \lim_{x\to 0} x^4 \sin\left(\frac{1}{3\sqrt{x}}\right) = 0$

31. (c) : $0 \le x^2 \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = x^2 \frac{1 - e^{-2/x}}{1 + e^{-2/x}} \le x^2$ for $x > 0$

So, $\lim_{x\to 0^+} f(x) = 0$

Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \frac{e^{2/x} - 1}{e^{2/x} + 1} = 0$

Hence $\lim_{x\to 0} f(x) = 0$

32. (a) : $\frac{b}{2} = \lim_{x\to 0} \frac{a\cos ax - 1 + \tan x}{2x}$

$\Rightarrow \quad b = \lim_{x\to 0} \frac{a\cos ax - 1}{x} + 1$

$\Rightarrow \quad a = b = 1$

33. (c) : $\lim_{x\to 0} \frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2} = 1$

implies $a - b = -1$ and in this case,

$1 = \lim_{x\to 0} \frac{(1 - \cos x) - ax\sin x}{3x^2} = \frac{1}{3}\left(\frac{1}{2} - a\right)$

This gives $a = -\frac{5}{2}$ and $b = a + 1 = -\frac{3}{2}$

34. (c) : $L = \lim_{x\to 0} \frac{\frac{x}{\sqrt{a^2 - x^2}} - x}{4x^3} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{a^2 - x^2}} - 1}{4x^2}$

$\Rightarrow \quad a = 1$ and $L = \lim_{x\to 0} \frac{1 - \sqrt{1 - x^2}}{4x^2} = \frac{1}{8}$

35. A → p, q, r; B → s; C → p, q, r; D → p, q, r, s

If $|x| > 1$, then $\lim_{n\to\infty} x^{2n} = \infty$, so

$f(x) = \lim_{n\to\infty} \frac{1 - x^{-2n}}{1 + x^{-2n}} = 1$

If $|x| < 1$ then $\lim_{n\to\infty} x^{2n} = 0$. therefore $f(x) = -1$. If $x = \pm 1$ then $x^{2n} = 1$ for any n, therefore $f(x) = 0$. Thus

$$f(x) = \begin{cases} 1 & \text{if } |x| > 1 \text{ i.e. } x \in (-\infty, -1) \cup (1, \infty) \\ -1 & \text{if } |x| < 1 \text{ i.e. } -1 < x < 1 \\ 0 & \text{if } x = \pm 1 \end{cases}$$

or $f(x) = \text{sgn}\,(|x| - 1)$

36. A → p; B → r; C → s; D → q

(A) For function f in (A)

$$f(x) = \begin{cases} x, & x > 1 \\ 0, & x = 1 \\ -x, & x < 1 \end{cases} \text{ so } \lim_{x\to 1} f(x) \text{ does not exist.}$$

(B) $\lim_{x\to 0} \frac{\sin(\sin(\tan(x^2/2)))}{\log\cos 3x}$

$= \lim_{x\to 0} \frac{\sin(\sin(\tan(x^2/2)))}{\sin(\tan x^2/2)}$

$\times \frac{\sin(\tan x^2/2)}{\tan x^2/2} \times \frac{\tan(x^2/2)}{x^2/2} \times \frac{x^2/2}{\log\cos 3x}$

$= \frac{1}{2} \lim_{x\to 0} \frac{2x}{-3\tan 3x} = \frac{-1}{9}$

(C) $\lim_{x\to 0} \frac{\sqrt[3]{1 + \tan^{-1} 3x} - \sqrt[3]{1 - \sin^{-1} 3x}}{\sqrt{1 - \sin^{-1} 2x} - \sqrt{1 + \tan^{-1} 2x}}$

$= \lim_{x\to 0} \frac{\left(1 + \frac{1}{3}\tan^{-1} 3x + \ldots\ldots\right) - \left(1 - \frac{1}{3}\sin^{-1} 3x + \ldots.\right)}{\left(1 - \frac{1}{2}\sin^{-1} 2x - \ldots\ldots\right) - \left(1 + \frac{1}{2}\tan^{-1} 2x + \ldots\ldots\right)}$

$= \frac{\frac{1}{3}\frac{\tan^{-1} 3x}{x} + \frac{1}{3}\frac{\sin^{-1} 3x}{x} + \frac{1}{x}}{-\frac{1}{2}\frac{\sin^{-1} 2x}{x} - \frac{1}{2}\frac{\tan^{-1} 2x}{x} + \frac{1}{x}}$

$= \frac{\text{(higher power of } \sin^{-1} 3x \text{ and } \tan^{-1} 3x)}{\text{(higher power of } \sin^{-1} 2x \text{ and } \tan^{-1} 2x)}$

$= \frac{1 + 1}{-1 - 1} = -1$

(D) $\lim_{x\to 0^-} \frac{e^{1/x} - 1}{e^{1/x} + 1} = -1$ and $\lim_{x\to 0^+} \frac{e^{1/x} - 1}{e^{1/x} + 1} = 1$

37. A → r; B → s; C → q, r; D → p

(A) By L' Hospital Rule

$\lim_{x\to 3} \frac{(x^3 + 27)\frac{1}{x - 2} + \log(x - 2).3x^2}{2x} = \frac{54}{6} = 9$

(B) $L = \lim_{x\to 0} \left(\frac{e^x - 1}{x}\right)^{\left(\frac{x}{x + 1 - e^x}\right)} = \lim_{x\to 0} \left(\frac{e^x - 1}{x}\right)^{\left(\frac{1}{1 - \frac{e^x - 1}{x}}\right)}$

Put $\frac{e^x - 1}{x} = t$, As $x \to 0, t \to 1$

$\therefore \quad L = \lim_{t\to 1} t^{\frac{1}{1 - t}}$ [1^∞ form]

Taking logarithm,

$$\log L = \lim_{t\to 1}\frac{1}{1-t}\log t \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{t\to 1}\frac{\frac{1}{t}}{-1} \text{ (by L Hospital's rule)}$$

$$= -1 \Rightarrow L = e^{-1}$$

(C) We know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.$$

$$\therefore\ L = \lim_{x\to 0}\frac{ax + x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.\right) - b\left(x - \frac{x^3}{3!} + \ldots.\right)}{x^3}$$

$$= \lim_{x\to 0}(a - b + 1)\times\frac{1}{x^2} + \left(\frac{b}{3!} - \frac{1}{2!}\right) + \text{terms containing } x$$

As $L = 1$, we must have

$a - b + 1 = 0$ and $\frac{b}{3!} - \frac{1}{2!} = 1$

$\Rightarrow\ b = 9$ and $a = 8$

(D) $L = \lim_{x\to 0}\frac{-f(x) + 3f(2x) - 3f(3x) + f(4x)}{x^3}\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{-f'(x) + 6f'(2x) - 9f'(3x) + 4f'(4x)}{3x^2}\quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-f''(x) + 12f''(2x) - 27f''(3x) + 16f''(4x)}{6x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-f'''(x) + 24f'''(2x) - 81f'''(3x) + 64f'''(4x)}{6}$$

$$= \frac{6.f'''(0)}{6} = f'''(0)$$

But $L = 12 \Rightarrow f'''(0) = 12$

38. (3) : Let $\lim_{n\to\infty} f(n) = l \Rightarrow \lim_{n\to\infty} f(n+1) = l$

$$\lim_{n\to\infty} f(n+1) = \frac{1}{2}\lim_{n\to\infty}\left[f(n) + \frac{9}{f(n)}\right]$$

$$\Rightarrow\quad l = \frac{1}{2}\left[l + \frac{9}{l}\right]$$

$$\Rightarrow\quad 2l = \frac{l^2 + 9}{l} \Rightarrow\ 2l^2 = l^2 + 9 \Rightarrow\ l^2 = 9 \Rightarrow\ l = \pm 3$$

$\because\ f(n) > 0\ \forall\ n \in N\ \ \therefore\ \lim_{n\to\infty} f(n) = 3$

39. (3) :

Let $\lim_{x\to 0}\frac{(\cos x - 1)(\cos x - e^x)}{x^n} = k$ (finite, non-zero)

$$\Rightarrow \lim_{x\to 0}\frac{\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots..\right) - 1\right]\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.\right) - \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.\right)\right]}{x^n} = k$$

As the limit is finite, non zero, we have degree of denominator = least power of x

$\Rightarrow\ n = 3$

40. (4) : $\lim_{x\to 0}\frac{ae^x - b\cos x + c.e^{-x}}{x\sin x} = 2 \Rightarrow a - b + c = 0$

Apply LH Rule

$$\lim_{x\to 0}\frac{ae^x + b\sin x - c.e^{-x}}{\sin x + x\cos x} = 2 \ \Rightarrow\ a + 0 - c = 0 \ \Rightarrow a = c$$

Apply LH rule

$$\lim_{x\to 0}\frac{ae^x + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x} = 2 \Rightarrow a + b + c = 4$$

$\therefore\ \ a + b + c = 4$

41. (1) : We have

$$\text{L.H.L.} = \lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{\tan x - \sin\tan^{-1}(\tan x)}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{\tan x - \sin x}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{1 - \frac{\sin x}{\tan x}}{1 + \frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1-0}{1+0}} = 1$$

At $x \to \frac{\pi}{2}^-$, $0 < x < \frac{\pi}{2}$ $\ \therefore\ \tan^{-1}(\tan x) = x$

Further as, $x \to \frac{\pi}{2}^-$, $\tan x \to \infty$ and $\cos^2(\tan x)$ is real number between 0 and 1]

$$\text{R.H.L.} = \lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x - \sin\tan^{-1}(\tan x)}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x+\sin x}{\tan x+\cos^2(\tan x)}}$$

$$= \lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{1+\frac{\sin x}{\tan x}}{1+\frac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1-0}{1-0}} = 1$$

As $x\to\frac{\pi}{2}^+, x>\frac{\pi}{2}$

$\Rightarrow$ $\tan^{-1}\tan x = \tan^{-1}\tan(x-\pi) = x-\pi$

$\therefore$ $\sin\tan^{-1}(\tan x) = \sin(x-\pi) = -\sin x$

(Further as $x\to\frac{\pi}{2}^+$; $\tan x\to-\infty$ and $\cos^2(\tan x)$ is a real number between 0 and 1)

$\therefore$ L.H.L. = R.H.L. = 1 $\therefore$ required limit = 1

42. (3) : Here two straight lines,

$(\tan\alpha)x + (\sin\alpha)y = \alpha$ and $(\alpha\ \text{cosec}\ \alpha)x + (\cos\alpha)y = 1$ have their point of intersection as,

$$x = \frac{\alpha\cos\alpha-\sin\alpha}{\sin\alpha-\alpha} \text{ and } y = \frac{\alpha-x\tan\alpha}{\sin\alpha}$$

$\therefore$ when $\alpha\to 0$, we obtain the point P.

i.e., $\lim_{\alpha\to0} x = \lim_{\alpha\to0}\frac{\alpha\cos\alpha-\sin\alpha}{\sin\alpha-\alpha}\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{\alpha\to0}\frac{-\alpha\sin\alpha+\cos\alpha-\cos\alpha}{\cos\alpha-1}$$

(applying L-Hospital's rule)

$$\lim_{\alpha\to0}\frac{-\alpha\sin\alpha}{-2\sin^2\alpha/2} = \lim_{\alpha\to0}\frac{\alpha\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}{2\sin^2\frac{\alpha}{2}}$$

$$= \lim_{\alpha\to0}\frac{\alpha}{\tan\alpha/2} = \lim_{\alpha\to0}\frac{2\frac{\alpha}{2}}{\tan\frac{\alpha}{2}} = 2$$

Again, $\lim_{\alpha\to0} y = \lim_{\alpha\to0}\frac{\alpha-x\tan\alpha}{\sin\alpha} = \lim_{x\to0}\left(\frac{\alpha}{\sin\alpha}-\frac{x}{\cos\alpha}\right)$

$$= \lim_{\alpha\to0}\frac{\alpha}{\sin\alpha} - \lim_{\alpha\to0}\frac{x}{\cos\alpha} = 1-2 = -1 \quad \left[\because \lim_{\alpha\to0} x = 2\right]$$

$$\Rightarrow \lim_{\alpha\to0} y = -1$$

Hence, in limiting position $P(2, -1)$

$\Rightarrow$ $h - k = 2 + 1 = 3$

43. (2) : $\because$ $f(x) = |x^2-3x+2| + \cos|x|$

$= |(x-1)|\,|(x-2)| + \cos|x|$

$$f(x) = \begin{cases} x^2-3x+2+\cos x, & x<0 \\ x^2-3x+2+\cos x, & 0\le x<1 \\ -x^2+3x-2+\cos x, & 1\le x<2 \\ x^2-3x+2+\cos x, & x>2 \end{cases}$$

$$\therefore\ f'(x) = \begin{cases} 2x-3-\sin x, & x<0 \\ 2x-3-\sin x, & 0\le x<1 \\ -2x+3-\sin x, & 1\le x<2 \\ 2x-3-\sin x, & x>2 \end{cases}$$

it is clear $f(x)$ is not differentiable at $x = 1$ and $x = 2$.

44. (7) : $1 = f(0) = \lim_{x\to\infty} f(x)$

$$= \lim_{x\to0}\frac{x\left(1+a\left(1-\frac{x^2}{\lfloor 2}+....\right)\right)-b\left(x-\frac{x^3}{\lfloor 3}+.....\right)}{x^3}$$

$$= \lim_{x\to0}\frac{x(1+a-b)+x^3\left(\frac{-a}{2}+\frac{b}{6}\right)+x^5(\lambda)+....}{x^3}$$

$\Rightarrow$ $1+a-b=0$ and $\frac{-a}{2}+\frac{b}{6}=1$

$\Rightarrow$ $a=\frac{-5}{2}, b=\frac{-3}{2}$ and $2a-8b=7$

45. (6) : $f(x+y) = \frac{f(2x)+f(2y)}{2}$

and $f(2x) = 2f(x) - 1$ (put $y = 0$)

Now $f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to0}\frac{f(2x)+f(2h)-2f(x)}{2h} = \lim_{h\to0}\frac{f(2h)-1}{2h}$$

$= f'(0) = -1$

Since $f(0) = 1$ and $f'(0) = -1$

$\therefore$ $f(x) = 1 - x \Rightarrow 5 - f(2) = 5 - (-1) = 6$

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WB JEE MOCK TEST PAPER

UNIT-III : CO-ORDINATE GEOMETRY

The entire syllabus of Mathematics of WB-JEE is being divided in to six units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issue.

CATEGORY-I

1. The circumcentre of a triangle with vertices (8, 6), (8, –2), (2, –2) is at the point
(a) (2, –1) (b) (1, –2) (c) (5, 2) (d) (2, 5)

2. If the co-ordinates of the circumcentre and the orthocentre of a triangle are respectively (5, 5) and (2, 2), then the co-ordinates of the centroid are
(a) (1, 1) (b) (3, 1) (c) (3, 3) (d) (4, 4)

3. If the points $A(3, 4)$, $B(x_1, y_1)$ and $C(x_2, y_2)$ are such that both 3, x_1, x_2 and 4, y_1, y_2 are in A.P., then
(a) A, B, C are vertices of an isosceles triangle
(b) A, B, C are collinear points
(c) A, B, C are vertices of a right angled triangle
(d) A, B, C are vertices of an equilateral triangle

4. If the point $(2\cos\theta, 2\sin\theta)$, for $\theta \in (0, 2\pi)$ lies in the region between the lines $x + y = 2$ and $x - y = 2$ containing origin, then θ lies in
(a) $\left(0, \frac{\pi}{2}\right) \cup \left(3\frac{\pi}{2}, 2\pi\right)$ (b) $[0, \pi]$
(c) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (d) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

5. Number of points having distance $\sqrt{5}$ from the straight line $x - 2y + 1 = 0$ and a distance $\sqrt{13}$ from the line $2x + 3y - 1 = 0$ are
(a) 1 (b) 2 (c) 4 (d) 5

6. If p and q are respectively the perpendiculars from the origin upon the straight lines whose equations are $x\sec\theta + y\,\text{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$, then $4p^2 + q^2$ is equal to
(a) $5a^2$ (b) $4a^2$ (c) $3a^2$ (d) a^2

7. The parametric form of equation of the circle $x^2 + y^2 - 6x + 2y - 28 = 0$ is
(a) $x = -3 + \sqrt{38}\cos\theta,\ y = -1 + \sqrt{38}\sin\theta$
(b) $x = \sqrt{38}\cos\theta,\ y = \sqrt{38}\sin\theta$
(c) $x = -3 - \sqrt{38}\cos\theta,\ y = -1 + \sqrt{38}\sin\theta$
(d) $x = 3 + \sqrt{38}\cos\theta,\ y = -1 + \sqrt{38}\sin\theta$

8. The line segment joining (5, 0) and $(10\cos\theta, 10\sin\theta)$ is divided internally in the ratio 2 : 3 at P. If θ varies, then the locus of P is
(a) a circle (b) a straight line
(c) a parabola (d) an ellipse

9. The shortest distance between the circles $(x - 1)^2 + (y + 2)^2 = 1$ and $(x + 2)^2 + (y - 2)^2 = 4$ is
(a) 1 (b) 2
(c) 3 (d) 4

10. The one end of the latus rectum of the parabola $y^2 - 4x - 2y - 3 = 0$ is at
(a) (0, –1) (b) (0, 1)
(c) (0, –3) (d) (3, 0)

11. If a point $P(x, y)$ moves along the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and C is the centre of the ellipse, then $4\max\{CP\} + 5\min\{CP\} =$
(a) 25 (b) 40
(c) 45 (d) 54

12. The length of the transverse axis of a hyperbola is $2\cos\alpha$. The foci of the hyperbola are the same as that of the ellipse $9x^2 + 16y^2 = 144$. The equation of the hyperbola is

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(a) $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{7-\cos^2\alpha}=1$

(b) $\frac{x^2}{\cos^2\alpha}-\frac{y^2}{7+\cos^2\alpha}=1$

(c) $\frac{x^2}{1+\cos^2\alpha}-\frac{y^2}{7-\cos^2\alpha}=1$

(d) $\frac{x^2}{1+\cos^2\alpha}-\frac{y^2}{7+\cos^2\alpha}=1$

13. If the co-ordinate axes are shifted to the point $(-1, 2)$ without rotation, then the curve whose equation is $2x^2 + y^2 - 4x + 4y = 0$ will have the equation

(a) $2x^2 + y^2 - 8x + 8y + 18 = 0$
(b) $2x^2 + y^2 - 6 = 0$
(c) $2x^2 + y^2 + 8y + 10 = 0$
(d) $x^2 + 2y^2 + 8x - 8y + 18 = 0$

14. If $x^2 + y^2 - 2x - 6y + 10 - r^2 = 0$ and $x^2 + y^2 + 6x = 0$ intersect in two distinct points, then

(a) $2 < r < 8$ (b) $r < 2$
(c) $r = 2$ (d) $r > 2$

15. Circles are drawn with SP as diameter where S is the focus of the parabola $y^2 + 4ax = 0$ and P is any point on the parabola. The locus of the centre of the circle is

(a) $y^2 = 2ax$ (b) $y^2 + 2ax + a^2 = 0$
(c) $y^2 + 2ax = 0$ (d) $y^2 + ax = 0$

16. If C is the centre A, B are two points on the conic $4x^2 + 9y^2 - 8x - 36y + 4 = 0$, such that $\widehat{ACB}=\frac{\pi}{2}$, then $CA^{-2} + CB^{-2} =$

(a) $\frac{13}{36}$ (b) $\frac{36}{13}$ (c) $\frac{16}{33}$ (d) $\frac{33}{16}$

17. The two conics $bx^2 = y$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intersect iff

(a) $-\frac{1}{\sqrt{2}}\le a\le\frac{1}{\sqrt{2}}$ (b) $a<-\frac{1}{\sqrt{2}}$

(c) $a>\frac{1}{\sqrt{2}}$ (d) $a < b$

18. If the straight lines $2x + 3y - 1 = 0$ and $x + 2y - 1 = 0$ and $ax + by - 1 = 0$ form a triangle with origin as orthocentre, then (a, b) is given by

(a) $(6, 4)$ (b) $(-3, 3)$
(c) $(-8, 8)$ (d) $(0, 7)$

19. If $5a + 4b + 20c = t$, then the value of t for which the line $ax + by + c - 1 = 0$ always passes through a fixed point is

(a) 0 (b) 20
(c) 30 (d) none of these

20. The line $3x + 4y - 24 = 0$ cuts the x-axis at A and y-axis at B. Then the incentre of the triangle OAB, where O is the origin is

(a) $(1, 2)$ (b) $(2, 2)$
(c) $(12, 12)$ (d) $(2, 12)$

21. If a line OP through the origin O makes angle α, $45°$ and $60°$ with x, y and z axes respectively, then the direction cosines of OP are

(a) $\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}$ (b) $\frac{1}{2},\frac{1}{2},\frac{1}{\sqrt{2}}$

(c) $\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2}$ (d) none of these

22. If O is the origin and the line OP of length r makes an angle α with x-axis and lies in the xz plane, then the co-ordinates of P are

(a) $(r\cos\alpha, 0, r\sin\alpha)$ (b) $(0, 0, r\sin\alpha)$
(c) $(0, 0, r\cos\alpha)$ (d) $(r\cos\alpha, 0, 0)$

23. If a straight line makes angles α, β, γ with the co-ordinate axes, then $\frac{1-\tan^2\alpha}{1+\tan^2\alpha}+\frac{1}{\sec 2\beta}-2\sin^2\gamma =$

(a) -1 (b) 1 (c) -2 (d) 2

24. The foot of the perpendicular from the point $(1, 6, 3)$ to the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ is

(a) $(1, 3, 5)$ (b) $(-1, -1, -1)$
(c) $(2, 5, 8)$ (d) $(-2, -3, -4)$

25. Equation of the straight line through the point $(2, 3, 1)$ and parallel to the line of intersection of the planes $x - 2y - z + 5 = 0$ and $x + y + 3z = 6$ is

(a) $\frac{x-2}{-5}=\frac{y-3}{-4}=\frac{z-1}{3}$ (b) $\frac{x-2}{5}=\frac{y-3}{-4}=\frac{z-1}{3}$

(c) $\frac{x-2}{4}=\frac{y-3}{3}=\frac{z-1}{2}$ (d) $\frac{x-2}{5}=\frac{y-3}{4}=\frac{z-1}{3}$

26. The lines

$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-K}$ and $\frac{x-1}{K}=\frac{y-4}{2}=\frac{z-5}{1}$

are coplanar if

(a) $K = 2$ (b) $K = 0$
(c) $K = 3$ (d) $K = -1$

27. The plane $x + 3y - 13 = 0$ passes through the line of intersection of the planes $2x - 8y + 4z + p = 0$ and $3x - 5y + 4z - 10 = 0$. If the plane is perpendicular to the plane $3x - y - 2z - 4 = 0$, then the value of p is equal to

(a) 2 (b) 5
(c) 9 (d) 3

28. Equation of the plane perpendicular to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and passing through the point (2, 3, 4) is

(a) $2x + 3y + z = 17$ (b) $x + 2y + 3z = 9$
(c) $3x + 2y + z = 16$ (d) $x + 2y + 3z = 20$

29. The line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is parallel to the plane

(a) $2x + 3y + 4z = 0$ (b) $3x + 4y + 5z = 7$
(c) $2x + y - 2z = 0$ (d) $x + y + z = 2$

30. The angle between two diagonals of a cube is

(a) $\cos^{-1}\left(\frac{1}{3}\right)$ (b) 30°
(c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) 45°

CATEGORY II

31. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is

(a) $y^2 = 2x$ (b) $x^2 = 2y$
(c) $x^2 = y$ (d) $y^2 = x$

32. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is

(a) 820 (b) 780
(c) 901 (d) 861

33. The equation of the plane containing the line $2x - 5y + z = 3$; $x + y + 4z = 5$, and parallel to the plane, $x + 3y + 6z = 1$ is

(a) $x + 3y + 6z = 7$ (b) $2x + 6y + 12z = -13$
(c) $2x + 6y + 12z = 13$ (d) $x + 3y + 6z = -7$

34. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 16$, is

(a) $3\sqrt{21}$ (b) 13
(c) $2\sqrt{14}$ (d) 8

35. If the abscissae and ordinates of two points P and Q are the roots of $x^2 + 2ax - b^2 = 0$ and $y^2 + 2py - q^2 = 0$, then the equation of the circle with PQ as diameter is

(a) $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$
(b) $x^2 + y^2 - 2ax - 2py + b^2 + q^2 = 0$
(c) $x^2 + y^2 - 2ax - 2py - b^2 - q^2 = 0$
(d) $x^2 + y^2 + 2ax + 2py + b^2 + q^2 = 0$

CATEGORY III

36. A circle S passes through the point (0, 1) and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then

(a) radius of S is 8
(b) radius of S is 7
(c) centre of S is (–7, 1)
(d) centre of S is (–8, 1)

37. The point P divides the line segment joining the points $A(-5, 1)$ and $B(3, 5)$ in $\lambda : 1$. The co-ordinates of the points Q and R are respectively (1, 5) and (7, 2). If the area of the triangle PQR is 2 square units, then the value of λ is

(a) $\frac{19}{5}$ (b) $\frac{31}{9}$
(c) 23 (d) 19

38. Let A and A' are the vertices of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose foci is S and S' and eccentricity e. Let P is a point on the ellipse such that $\frac{ar(\Delta AA'P)}{ar(\Delta SS'P)} = K$, then

(a) $0 < K < 1$
(b) $K > 1$
(c) K, is only dependent on e
(d) K, is only dependent on b and e.

39. The direction line makes an angle 60° and 45° with the axes of x and y respectively. The angle it makes with the axis of z is

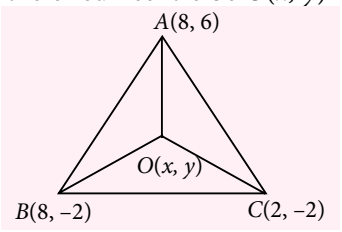
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

40. The length and foot of perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$ are

(a) $3\sqrt{21}$ (b) $\sqrt{63}$
(c) (1, 2, 8) (d) (2, 4, –1)

SOLUTIONS

1. (c) : Let the circumcentre be $O(x, y)$



$\therefore \quad \overline{OA} = \sqrt{(x-8)^2 + (y-6)^2}$

$\overline{OB} = \sqrt{(x-8)^2 + (y+2)^2}$

$\overline{OC} = \sqrt{(x-2)^2 + (y+2)^2}$

Clearly $\overline{OA} = \overline{OB} = \overline{OC}$

Now $\overline{OA} = \overline{OB}$ gives $(y-6)^2 = (y+2)^2$

$\Rightarrow \quad y^2 - 12y + 36 = y^2 + 4y + 4$

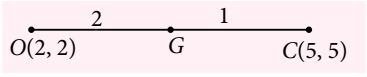
$\Rightarrow \quad 16y = 32 \quad \therefore \quad y = 2$

and $\overline{OB} = \overline{OC}$ gives $x^2 - 16x + 64 = x^2 - 4x + 4$

$\Rightarrow \quad 12x = 60 \quad \therefore \quad x = 5$

Thus circumcentre is (5, 2)

2. (d) :



We know that centroid (G) divides the line segment joining the orthocentre (O) and circumcentre (C) in 2 : 1.

$\therefore \quad G = \left(\dfrac{(2\times 5) + (1\times 2)}{2+1}, \dfrac{(2\times 5) + (1\times 2)}{3} \right)$

$= (4, 4)$

3. (b) : We have $x_1 - 3 = x_2 - x_1$ and $y_1 - 4 = y_2 - y_1$

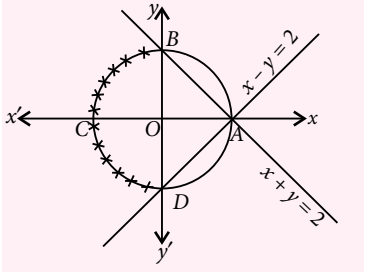
[As both 3, x_1, x_2 and 4, y_1, y_2 are in A.P]

$\Rightarrow \quad \dfrac{y_1 - 4}{x_1 - 3} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$\therefore$ Slope of AB = slope of BC

$\therefore$ A, B, C are collinear.

4. (c) :



Clearly $(2\cos\theta, 2\sin\theta)$ is a point on the circle $x^2 + y^2 = 2^2$

And the region between the lines $x + y = 2$ and $x - y = 2$ containing origin is $ABCDA$

$\therefore$ All the points are lying on BCD of the circle for which $\theta \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2} \right)$

5. (c) : By the problem, we have

$$\frac{|x - 2y + 1|}{\sqrt{1^2 + (-2)^2}} = \sqrt{5} \text{ and } \frac{|2x + 3y - 1|}{\sqrt{2^2 + 3^2}} = \sqrt{13}$$

$\Rightarrow \quad x - 2y + 1 = \pm 5$ and $2x + 3y - 1 = \pm 13$

Clearly, on solving the above equations, we will get 4 points.

6. (d) : We have $p = \dfrac{|-a|}{\sqrt{\sec^2\theta + \text{cosec}^2\theta}}$

and $q = \dfrac{|-a\cos 2\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}}$

$\Rightarrow \quad p^2 = \dfrac{a^2}{\sec^2\theta + \text{cosec}^2\theta}$ and $q^2 = \dfrac{a^2\cos^2 2\theta}{1}$

$\Rightarrow \quad p^2 = \dfrac{a^2}{4}\sin^2 2\theta$ and $q^2 = a^2\cos^2 2\theta$

$\therefore \quad 4p^2 + q^2 = a^2$

7. (d) : The given equation of the circle is

$x^2 + y^2 - 6x + 2y - 28 = 0$

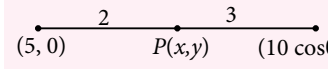
$\therefore$ centre $(3, -1)$ and radius $= \sqrt{3^2 + 1^2 + 28} = \sqrt{38}$

$\therefore \quad x = h + r\cos\theta$, $y = k + r\sin\theta$ are parametric form.

$\Rightarrow \quad x = 3 + \sqrt{38}\cos\theta$, $y = -1 + \sqrt{38}\sin\theta$

[where $(h, k) \equiv (3, -1)$ and $r = \sqrt{38}$]

8. (a) :



Using section formula, we get

$$x = \frac{20\cos\theta + 15}{5} \text{ and } y = \frac{20\sin\theta + 0}{5}$$

$\Rightarrow \quad 5x - 15 = 20\cos\theta$ and $5y = 20\sin\theta$

$\Rightarrow \quad \dfrac{x-3}{4} = \cos\theta$ and $\dfrac{y}{4} = \sin\theta$

$\because \quad \sin^2\theta + \cos^2\theta = 1$

$\Rightarrow$ $(x - 3)^2 + y^2 = 16$ is the required locus representing an equation of a circle.

9. (b) : Centre and radius of the circle $(x - 1)^2 + (y + 2)^2 = 1$ is $C_1(1, -2)$ and $r_1 = 1$.

Again centre and radius of the circle $(x + 2)^2 + (y - 2)^2 = 4$ is $C_2(-2, 2)$ and $r_2 = 2$

Now $C_1C_2 = \sqrt{(1+2)^2 + (-2-2)^2} = \sqrt{9+16} = 5$

Since $C_1C_2 > r_1 + r_2$

$\therefore$ Two circles do not intersect.

$d_{min} = C_1C_2 - (r_1 + r_2) = 5 - (1 + 2) = 2$

10. (a) : The given equation of the parabola is

$y^2 - 4x - 2y - 3 = 0$

$\Rightarrow (y^2 - 2y + 1) = 4x + 4$

$\Rightarrow (y - 1)^2 = 4(x + 1)$

Let $Y = y - 1$ and $X = x + 1$

$\therefore Y^2 = 4X = 4 \cdot 1 \cdot X \quad \therefore a = 1$

End points of latus rectum are $(a, \pm 2a) = (1, \pm 2)$

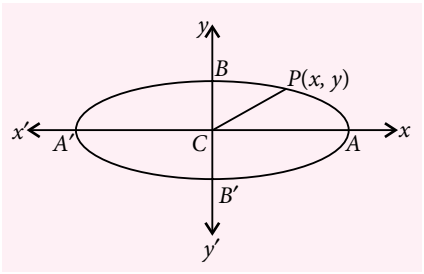
$\therefore X = 1 \Rightarrow x + 1 = 1 \Rightarrow x = 0$

$Y = \pm 2 \Rightarrow y - 1 = \pm 2 \Rightarrow y = 3, -1$

$\therefore$ End points of latus rectum are (0, 3) and (0, –1).

11. (b) : The equation is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$



$\therefore a = 5, b = 4$

Clearly max $\{CP\}$ = Length of semi-major axis = 5

and min $\{CP\}$ = Length of semi-minor axis = 4

$\therefore$ 4max $\{CP\}$ + 5min$\{CP\}$ = $(4 \times 5) + (5 \times 4) = 40$

12. (a) : Length of transverse axis of hyperbola $= 2\cos\alpha$

$\therefore 2a = 2\cos\alpha \Rightarrow a = \cos\alpha$ ...(i)

Now the equation of the given ellipse is

$9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$$

$\therefore$ foci of ellipse $= (\pm\sqrt{7}, 0)$ = foci of hyperbola

$$\Rightarrow e\cos\alpha = \pm\sqrt{7} \Rightarrow e = \pm\frac{\sqrt{7}}{\cos\alpha} \quad \text{...(ii)}$$

Since for hyperbola, $e^2 = \frac{b^2}{a^2} + 1$

$$\Rightarrow \left(\pm\frac{\sqrt{7}}{\cos\alpha}\right)^2 = \frac{b^2}{\cos^2\alpha} + 1 \Rightarrow \frac{7}{\cos^2\alpha} = \frac{b^2}{\cos^2\alpha} + 1$$

$\Rightarrow 7 = b^2 + \cos^2\alpha \Rightarrow b^2 = 7 - \cos^2\alpha$

$\therefore$ Required equation of hyperbola is

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{7 - \cos^2\alpha} = 1$$

13. (a) : Here, the origin has been shifted to (–1, 2)

$\therefore$ by translation, we have

$x = x' - 1$ and $y = y' + 2$

The equation of the curve becomes

$2(x' - 1)^2 + (y' + 2)^2 - 4(x' - 1) + 4(y' + 2) = 0$

$\Rightarrow 2x'^2 + y'^2 - 8x' + 8y' + 18 = 0$

14. (a) : The given two equations of circles are

$x^2 + y^2 - 2x - 6y + 10 - r^2 = 0$

and $x^2 + y^2 + 6x = 0$

Let they be denoted by C_1 and C_2 respectively.

Now $C_1 : (x - 1)^2 + (y - 3)^2 = r^2$

and $C_2 : (x + 3)^2 + y^2 = 3^2$

$\therefore$ Centre of C_1 is (1, 3) and that of C_2 is (–3, 0)

Let the distance between the centres of the above circles be d

$$\therefore d = \sqrt{(1+3)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

For two circles to intersect, $r_1 - r_2 < d < r_1 + r_2$

$\Rightarrow r - 3 < 5 < r + 3 \Rightarrow 2 < r < 8$

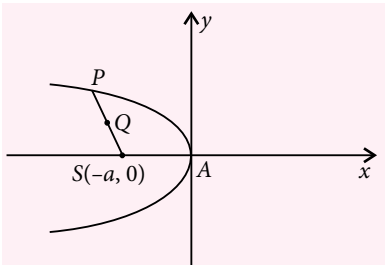
15. (b) : The equation of the parabola is $y^2 + 4ax = 0$

$\Rightarrow y^2 = -4ax$

$\therefore$ focus $(-a, 0)$

Let $P(-at^2, 2at)$ be any point on the parabola.

For the circle with SP as diameter, the centre Q is the mid-point of SP.



If $Q = (x, y)$ then

$$x = \frac{-at^2 - a}{2} \Rightarrow 2x = -at^2 - a \quad \text{...(i)}$$

$$\text{and } y = \frac{2at + 0}{2} \Rightarrow 2y = 2at \Rightarrow y = at \quad \text{...(ii)}$$

Eliminating t, from (i) and (ii), we get

$$2x = -a\left(\frac{y}{a}\right)^2 - a = -a\frac{y^2}{a^2} - a = -\frac{y^2}{a} - a$$

$\Rightarrow 2ax = -y^2 - a^2 \Rightarrow y^2 + 2ax + a^2 = 0$ is required equation of locus.

16. (a) : The given equation of the conic is

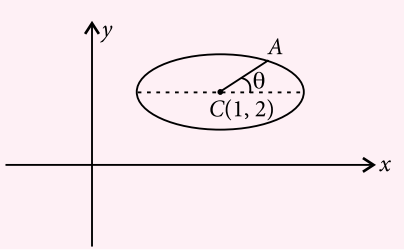
$4x^2 + 9y^2 - 8x - 36y + 4 = 0$

$\Rightarrow 4(x^2 - 2x) + 9(y^2 - 4y) + 4 = 0$

$\Rightarrow 4(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 4 = 4 + 36$

$\Rightarrow 4(x - 1)^2 + 9(y - 2)^2 = 36,$

which is an ellipse whose centre is (1, 2)



Let CA makes an angle θ with the major axis, then

$A = (1 + CA\cos\theta, 2 + CA\sin\theta)$

Similarly, $B = \left[1 + CB\cos\left(\frac{\pi}{2} + \theta\right), 2 + CB\sin\left(\frac{\pi}{2} + \theta\right)\right]$

As A and B are points on the conic

$CA^2(4\cos^2\theta + 9\sin^2\theta) = 36$

and $CB^2(4\sin^2\theta + 9\cos^2\theta) = 36$

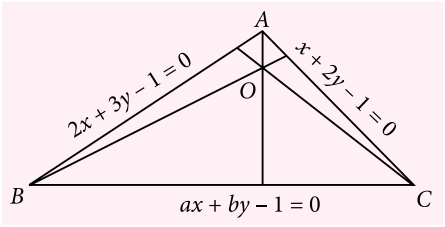
$\therefore \quad CA^{-2} + CB^{-2} = \frac{13}{36}$

17. (a) : Substituting $x^2 = \frac{1}{b}y$, the equation of the conic gives the equation $a^2y^2 - by + a^2b^2 = 0$

This has real roots iff $b^2 - 4a^4b^2 \geq 0$

i.e. $a^4 \leq \frac{1}{4}$ giving $-\frac{1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$

18. (c) :



Equation of AO is

$2x + 3y - 1 + \lambda(x + 2y - 1) = 0$

Since the above line is passing through origin

$\therefore \quad \lambda = -1$

$\therefore$ equation of AO is $x + y = 0$

Again $AO \perp BC$

$\therefore \quad -1 \times \left(\frac{-a}{b}\right) = -1 \quad \Rightarrow \quad a = -b$

Similarly $(2x + 3y - 1) + \mu(ax - ay - 1) = 0$ will be the equation of BO for $\mu = -1$

$$BO \perp AC \Rightarrow -\frac{(2-a)}{3+a} \times -\frac{1}{2} = -1$$

$$\Rightarrow \quad \frac{2-a}{2(3+a)} = -1$$

$\Rightarrow \quad 6 + 2a = -2 + a \quad \Rightarrow \quad a = -8$

$\Rightarrow \quad a = -8$ and $b = 8$

19. (b) : Equation $\frac{ax}{c-1} + \frac{by}{c-1} + 1 = 0$ has two independent parameters. It can pass through a fixed point if it contains only one independent parameter. Now there must be one relation between $\frac{a}{c-1}$ and $\frac{b}{c-1}$ independent of a, b and c.

So that $\frac{a}{c-1}$ can be expressed in terms of $\frac{b}{c-1}$ and straight line contains only one independent parameter.

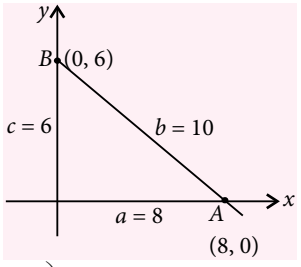
Now that given relation can be expressed as

$$\frac{5a}{c-1} + \frac{4b}{c-1} = \frac{t - 20c}{c-1}$$

Now R.H.S. be independent of c if $t = 20$

20. (b) : The line $3x + 4y = 24$

i.e. $\frac{x}{8} + \frac{y}{6} = 1$



meets the x-axis at $A(8, 0)$ and $B(0, 6)$

$\Rightarrow \quad AB = 10$

The co-ordinates of the incentre of ΔABC are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$$

$\therefore$ For ΔOAB, I is

$$\left(\frac{8(0) + 10(0) + 6(8)}{8+10+6}, \frac{8(6) + 10(0) + 6(0)}{8+10+6}\right)$$ *i.e.*, I is $(2, 2)$

21. (c) : The direction cosines of OP are $\cos\alpha$, $\cos45°$, $\cos60°$

Now $\cos^2\alpha + \cos^245° + \cos^260° = 1$

$$\Rightarrow \quad \cos^2\alpha + \frac{1}{2} + \frac{1}{4} = 1 \quad \Rightarrow \quad \cos^2\alpha = \frac{1}{4}$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2}$$

Thus, the direction cosines of OP are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

22. (a) : Let the coordinates of P be (x, y, z).

Since OP lies in xz plane and makes an angle α with the x-axis, it makes angle $\left(\frac{\pi}{2} - \alpha\right)$ with z axis and $\frac{\pi}{2}$ with y-axis.

So, $x = r\cos\alpha$, $y = r\cos\frac{\pi}{2} = 0$, $z = r\cos\left(\frac{\pi}{2} - \alpha\right)$

are the required co-ordinates and therefore co-ordinates of $P = (r\cos\alpha, 0, r\sin\alpha)$

23. (c) : $\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}+\dfrac{1}{\sec 2\beta}-2\sin^2\gamma$

$= \cos2\alpha + \cos2\beta + 1 - 2\sin^2\gamma - 1$

$= \cos2\alpha + \cos2\beta + \cos2\gamma - 1$

$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 - 1$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 4$

$= 2\cdot1 - 4 \quad [\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1]$

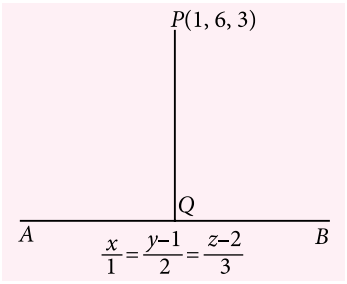
$= -2$

24. (a) : Let Q be the foot of $\perp r$ of PQ on the line

$\dfrac{x}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{3}$

Let $\dfrac{x}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{3}=\lambda$ (say)

$\therefore \quad x = \lambda,\ y = 2\lambda + 1,\ z = 3\lambda + 2$



$\therefore$ The point $Q = (\lambda, 2\lambda + 1, 3\lambda + 2)$ lie on AB.

Since $PQ \perp AB$

$\therefore \quad (\lambda - 1)\cdot1 + (2\lambda + 1 - 6)\cdot2 + (3\lambda + 2 - 3)\cdot3 = 0$

$\Rightarrow \quad \lambda - 1 + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$

$\Rightarrow \quad 14\lambda - 14 = 0 \quad \Rightarrow \quad \lambda = 1$

$\therefore$ required point $Q = (1, 3, 5)$

25. (a) : Let the direction ratios of the line of intersection of the planes be $<a, b, c>$

$\therefore \quad a - 2b - c = 0$ and $a + b + 3c = 0$

Solving the above equations, we get

$\dfrac{a}{-5}=\dfrac{b}{-4}=\dfrac{c}{3}=k$ (say)

$\therefore \quad a = -5k,\ b = -4k,\ c = 3k$

$\Rightarrow \quad \dfrac{x-2}{-5}=\dfrac{y-3}{-4}=\dfrac{z-1}{3}$ is required equation of straight line.

26. (b) : The given lines are

$\dfrac{x-2}{1}=\dfrac{y-3}{1}=\dfrac{z-4}{-K}$ ...(i)

and $\dfrac{x-1}{K}=\dfrac{y-4}{2}=\dfrac{z-5}{1}$ ...(ii)

Now, (i) and (ii) are coplanar

$\therefore \quad \begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & -K \\ K & 2 & 1 \end{vmatrix} = 0$

On simplification, we get $K = 0$, or $K = -3$

27. (d) : The equation of the plane, passing through the line of intersection of the planes

$2x - 8y + 4z + p = 0$ and $3x - 5y + 4z - 10 = 0$ is

$(2x - 8y + 4z + p) + \lambda(3x - 5y + 4z - 10) = 0$

$\Rightarrow \quad (2 + 3\lambda)x - (8 + 5\lambda)y + 4(1 + \lambda)z + p - 10\lambda = 0$ ...(i)

Since (i) is perpendicular to $3x - y - 2z - 4 = 0$

$\therefore \quad 3(2 + 3\lambda) + (8 + 5\lambda) - 2(4 + 4\lambda) = 0$

$\Rightarrow \quad \lambda = -1$

Thus (i) becomes $-x - 3y + p + 10 = 0$

$\Rightarrow \quad x + 3y - (p + 10) = 0$ ...(ii)

$\therefore$ equation (ii) is identical with $x + 3y - 13 = 0$

$\therefore \quad p + 10 = 13 \quad \Rightarrow \quad p = 3$

28. (d) : Let the equation of plane be,

$ax + by + cz + d = 0$ ...(i)

Now the given line is

$\dfrac{x-0}{1}=\dfrac{y-0}{2}=\dfrac{z-3}{3}$ ...(ii)

Also, plane (i) is perpendicular to the line (ii).

So, $\dfrac{a}{1}=\dfrac{b}{2}=\dfrac{c}{3}=\lambda$

$\Rightarrow \quad a = \lambda,\ b = 2\lambda,\ c = 3\lambda$ ...(iii)

Also, plane (i) passes through (2, 3, 4).

So $2a + 3b + 4c + d = 0$ ...(iv)

From (iii) and (iv), we have

$2\lambda + 6\lambda + 12\lambda + d = 0 \quad \Rightarrow \quad d = -20\lambda$

$\therefore$ Required equation of plane is,

$\lambda x + 2\lambda y + 3\lambda z - 20\lambda = 0$

$\Rightarrow \quad x + 2y + 3z = 20$

29. (b) : Let the equation of plane be

$ax + by + cz + d = 0$ ...(i)

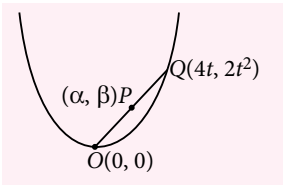
The given line $\dfrac{x-2}{3}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$ is parallel to plane (i)

So, $3a + 4b + 5c = 0$

Now from the option only plane in (b) satisfied the equation. So, (b) is the correct option.

30. (a) : The angle between two diagonals of a cube is $\cos^{-1}\left(\dfrac{1}{3}\right)$

31. (b) :



$OP : PQ = 1 : 3$

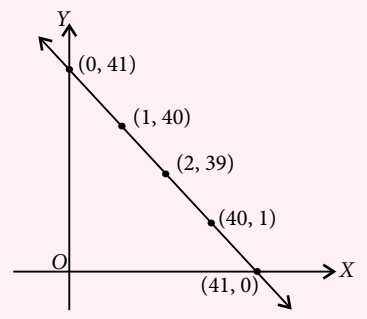
Let the co-ordinates of Q be $(4t, 2t^2)$

$\therefore$ By section formula, we get

$$\alpha = \frac{4t+0}{4} = t \text{ and } \beta = \frac{2t^2+0}{4} = \frac{t^2}{2}$$

Eliminating 't', we get the locus of $P(\alpha, \beta)$ as $\alpha^2 = 2\beta$
Thus the locus is $x^2 = 2y$.

32. (b) : We count the number of points on line $x = n$, $1 \le n < 40$, that lie in the interior of the triangle.



At line, for $x = 40$, we have 0 point, for $x = 39$, we have 1 point .....
$x = n$, we have $(40 - n)$ points
The total number of points = 1 + 2 + 3 + ... 39

$$= \frac{1}{2} \times 39 \times 40 = 39 \times 20 = 780$$

33. (a) : The equation of the plane containing the line $2x - 5y + z = 3$ and $x + y + 4z = 5$ is

$(2x - 5y + z - 3) + \lambda (x + y + 4z - 5) = 0$
$\Rightarrow (2 + \lambda)x + (\lambda - 5)y + (4\lambda + 1)z - (5\lambda + 3) = 0$
As the plane is parallel to $x + 3y + 6z - 1 = 0$,
The coefficient must be proportional giving

$$\frac{2+\lambda}{1} = \frac{\lambda-5}{3} = \frac{4\lambda+1}{6}$$

Taking any two of them give,

$6 + 3\lambda = \lambda - 5 \Rightarrow 2\lambda = -11$

$$\Rightarrow \lambda = -\frac{11}{2}$$

The required equation of plane is

$$\frac{-7x}{2} - \frac{21y}{2} - 21z + \frac{49}{2} = 0$$

i.e. $7x + 21y + 42z - 49 = 0$
i.e. $x + 3y + 6z = 7$

34. (b) : Let the parameter corresponding to the point of intersection be denoted by t, then

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = t$$

Thus $(3t + 2, 4t - 1, 12t + 2)$ is a general point.
This point lies on given plane $x - y + z = 16$ gives
$(3t + 2) - (4t - 1) + (12t + 2) = 16$
$\Rightarrow 11t = 11$
$\therefore t = 1$

Thus the point is (5, 3, 14)
Given point is (1, 0, 2)
The distance between the points

$$= \sqrt{(5-1)^2 + (3-0)^2 + (14-2)^2}$$

$$\Rightarrow \sqrt{16 + 9 + 144}$$

$$\Rightarrow \sqrt{169} = 13$$

35. (a) : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$
$\therefore x_1 + x_2 = -2a$ and $x_1x_2 = -b^2$
$y_2 + y_2 = -2p$ and $y_1y_2 = -q^2$
Equation of the required circle is

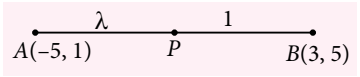
$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
$\Rightarrow x^2 - (x_1 + x_2)x + x_1x_2 + y^2 - (y_1 + y_2)y + y_1y_2 = 0$
$\Rightarrow x^2 + 2ax - b^2 + y^2 + 2py - q^2 = 0$
$\Rightarrow x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

36. (b, c,) : Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be orthogonal to $x^2 + y^2 - 2x - 15 = 0$ and $x^2 + y^2 - 1 = 0$.
Orthogonal condition requires

$-2g = c - 15$ and $c - 1 = 0$
Also (0, 1) lies on the circle $\Rightarrow 1 + 2f + c = 0$
Solving, we get $c = 1$, $g = 7$, $f = -1$

$$\text{Hence radius} = \sqrt{g^2 + f^2 - c} = \sqrt{7^2 + 1^2 - 1^2} = 7$$

37. (a, c) : Let the co-ordinates of P be (x, y)



$$\therefore x = \frac{3\lambda - 5}{\lambda + 1} \text{ and } y = \frac{5\lambda + 1}{\lambda + 1}$$

$\therefore$ The vertices of ΔPQR are

$$P\left(\frac{3\lambda-5}{\lambda+1}, \frac{5\lambda+1}{\lambda+1}\right),\ Q(1,5) \text{ and } R(7,2)$$

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Now, area of the triangle PQR is

$$\frac{1}{2}\left\{\frac{3\lambda-5}{\lambda+1}(5-2)+1\left(2-\frac{5\lambda+1}{\lambda+1}\right)+7\left(\frac{5\lambda+1}{\lambda+1}-5\right)\right\}$$

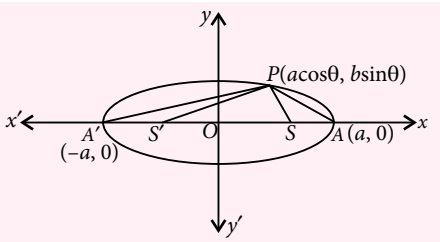
$$=\frac{3\lambda-21}{\lambda+1} \quad \text{[on simplification]}$$

$\therefore$ By the problem, $\left|\frac{3\lambda-21}{\lambda+1}\right|=2$

$\Rightarrow \quad \lambda=23$ and $\lambda=\frac{19}{5}$

38. (b, c) : The given equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$S \equiv (ae, 0)$ and $S' \equiv (-ae, 0)$

$\therefore \quad \frac{ar(\Delta AA'P)}{ar(\Delta SS'P)}=\frac{ab\sin\theta}{abe\sin\theta}=\frac{1}{e}$

- $K > 1$ as $e < 1$
- K is only dependent on e.

39. (b , c) : Let the direction line makes an angle α with z axis

$\therefore \quad \cos^2 60° + \cos^2 45° + \cos^2\alpha = 1$

$\Rightarrow \quad \cos^2\alpha = 1-\frac{3}{4}=\frac{1}{4} \quad \therefore \cos\alpha=\pm\frac{1}{2}$

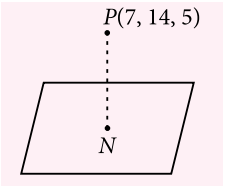
$\Rightarrow \quad \alpha=\frac{\pi}{3} \text{ or } \frac{2\pi}{3}$

40. (a, c) : The given plane is

$2x + 4y - z = 2$ ...(i)

Direction ratios of any line normal to (i) are

<2, 4, −1>



Hence, the equations of the line through the given point (7, 14, 5) and perpendicular to the plane (i) are

$$\frac{x-7}{2}=\frac{y-14}{4}=\frac{z-5}{-1} \quad \text{...(ii)}$$

Any point on the line (ii) is $N(2\lambda + 7, 4\lambda + 14, -\lambda + 5)$

It lies on plane (i)

$\therefore \quad 2(2\lambda + 7) + 4(4\lambda + 14) - 1(-\lambda + 5) = 2$

$\Rightarrow \quad 21\lambda + 65 = 2 \quad \Rightarrow \quad 21\lambda = -63$

$\Rightarrow \quad \lambda = -3$

$\therefore$ The foot of the perpendicular from P to the given plane is $N(2 \times (-3) + 7, 4(-3) + 14, -(-3) + 5)$

i.e. $N(1, 2, 8)$

$\therefore$ the length of perpendicular from (7, 14, 5) to the plane (i) is $|NP|$

$$= \sqrt{(7-1)^2+(14-2)^2+(5-8)^2}$$

$$= \sqrt{36+144+9} = 3\sqrt{21}$$

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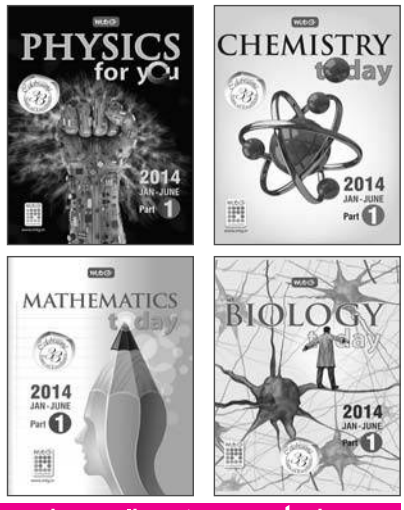

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Series 4

CONTINUITY & DIFFERENTIABILITY

CONTINUITY

A function $f(x)$ will be continuous at a point $x = a$, if there is no break or cut or hole or gap in the graph of the function $y = f(x)$ at the point $(a, f(a))$. Otherwise, it is discontinuous at that point.

A function f is said to be continuous at the point $x = a$ if the following conditions are satisfied :

(i) $f(a)$ exists.

(ii) $\lim\limits_{x \to a} f(x)$ exists.

(iii) $\lim\limits_{x \to a} f(x) = f(a)$.

CONTINUITY OF A FUNCTION ON AN INTERVAL

Continuity on an Open Interval

A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at each point of (a, b).

Continuity on a Closed Interval

A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ if

(i) $f(x)$ is continuous from right at $x = a$, *i.e.* $\lim\limits_{h \to 0} f(a + h) = f(a)$

(ii) $f(x)$ is continuous from left at $x = b$, *i.e.* $\lim\limits_{h \to 0} f(b - h) = f(b)$

(iii) $f(x)$ is continuous at each point of the open interval (a, b).

DISCONTINUITY OF A FUNCTION

1.	At a point	A real valued function $f(x)$ is said to be discontinuous at $x = a$, if it is not continuous at $x = a$. The discontinuity may be due to any of the following reasons: (i) $\lim\limits_{x \to a^-} f(x)$ or $\lim\limits_{x \to a^+} f(x)$ or both may not exist. (ii) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist but are not equal. (iii) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist and are equal but both may not be equal to $f(a)$.
2.	In an interval	A real valued function $f(x)$ is said to be discontinuous if it is not continuous at atleast one point in the given interval.

TYPES OF DISCONTINUITY

1.	Removable discontinuity	$f(x)$ is said to have a removable discontinuity at $x = a$ if $\lim\limits_{x \to a} f(x)$ exists but $\lim\limits_{x \to a^+} f(x) \neq f(a)$ or if and only if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) \neq f(a)$
2.	Non-removable discontinuity	$f(x)$ is said to have non-removabole discontinuity at $x = a$; (i) First kind: If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$ (ii) Second kind: If $\lim\limits_{x \to a^-} f(x)$ or $\lim\limits_{x \to a^+} f(x)$ or both do not exist.

ALGEBRA OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be two continuous functions on their common domain D and let c be a real number. Then

(i) $f + g$ is continuous at $x = c$

(ii) $f - g$ is continuous at $x = c$

(iii) fg is continuous at $x = c$

(iv) $\frac{f}{g}$ is continuous at $x = c$

Note :

- If f and g are real functions such that fog is defined and if g is continuous at a point a and f is continuous at $g(a)$, then fog is continuous at $x = a$.

DIFFERENTIABILITY

Let $f(x)$ be a real function and a be any real number. Then, we define

(i) **Right-hand derivative :** $\lim\limits_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$, and is denoted by $Rf'(a)$.

(ii) **Left-hand derivative :** $\lim\limits_{h \to 0^-} \frac{f(a-h) - f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$, and is denoted by $Lf'(a)$.

A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.

The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and it is known as the derivative of $f(x)$ at $x = a$.

If, however, $Rf'(a) \neq Lf'(a)$ we say that $f(x)$ is not differentiable at $x = a$.

Note :

- $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point.
- If a function is differentiable at a point, then it is necessarily continuous at that point. But the converse is not necessarily true.
- A function f is said to be a differentiable function if it is differentiable at every point in its domain.

DERIVATIVE OF A FUNCTION

If a function $f(x)$ is differentiable at every point in its domain, then

$\lim\limits_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$ or $\lim\limits_{h \to 0^-} \frac{f(x-h) - f(x)}{-h}$ is called the derivative or differentiation of f at x and is denoted by $f'(x)$ or $\frac{d}{dx} f(x)$.

SOME PROPERTIES OF DERIVATIVE

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
6.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$
7.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
8.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.

SOME GENERAL DERIVATIVES

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}};\ x \in (-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}};\ x \in (-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2};\ x \in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2};\ x \in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x \in R-[-1,1]$	$\operatorname{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}};\ x \in R-[-1,1]$
$\log_e x$	$\frac{1}{x};\ x > 0$	a^x	$a^x \log_e a;\ a > 0$	$\log_a x$	$\frac{1}{x \log_e a};\ x > 0$ and $a > 0$,

SOME IMPORTANT THEOREMS

Rolle's theorem

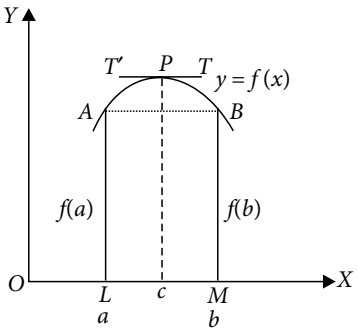
If a function $f(x)$ is

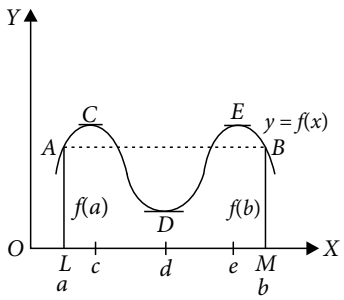
(i) continuous in the closed interval $[a, b]$ *i.e.* continuous at each point in the interval $[a, b]$

(ii) differentiable in an open interval (a, b) *i.e.* differentiable at each point in the open interval (a, b) and

(iii) $f(a) = f(b)$, then there will be at least one point c, in the interval (a, b) such that $f'(c) = 0$.

Geometrical meaning of Rolle's theorem





If the graph of a function $y = f(x)$ be continuous at each point from the point $A(a, f(a))$ to the point $B(b, f(b))$ and tangent to the graph at each point between A and B is unique *i.e.* tangent at each point between A and B exists and ordinates *i.e.* y co-ordinates of points A and B are equal, then there will be at least one point P on the curve between A and B at which tangent will be parallel to x-axis.

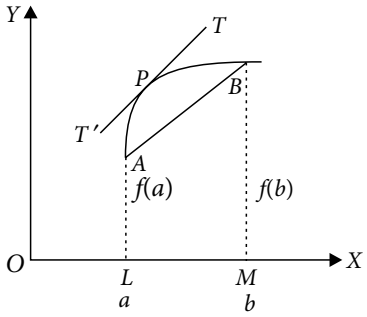
Lagrange's mean value theorem

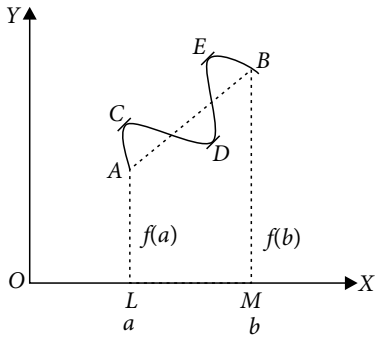
If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$ *i.e.* continuous at each point in the interval $[a, b]$

(ii) differentiable in the open interval (a, b) *i.e.* differentiable at each point in the interval (a, b) then there will be at least one point c, where $a < c < b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Geometrical meaning of Lagrange's mean value theorem





If the graph of a function $y = f(x)$ be continuous at each point from the point $A\ (a, f(a))$ to the point $B\ (b, f(b))$ and tangent at each point between A and B exists *i.e.* tangent is unique then there will be at least one point P on the curve between A and B, where tangent will be parallel to chord AB.

Very Short Answer Type

1. Discuss the continuity of the function $f(x) = \sin x - \cos x$

2. Differentiate $\cos(\sin x)$ with respect to x.

3. If $xy = x^3 + y^3$, find $\dfrac{dy}{dx}$.

4. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

5. Is the function defined by

$$f(x) = \begin{cases} x, \text{if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}$$ continuous at $x = 1$?

Short Answer Type

6. If the function $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, \text{ if } x \ne \dfrac{\pi}{2} \\ 3, \qquad \text{if } x = \dfrac{\pi}{2} \end{cases}$

is continuous at $x = \dfrac{\pi}{2}$, then find the value of k.

7. If $y^x = e^{y-x}$, then find the value of $\dfrac{dy}{dx}$.

8. Show that $f(x) = [x]$ is not differentiable at $x = 1$.

9. If $y = \sqrt{e^{\sqrt{x}}}$, find $\dfrac{dy}{dx}$.

10. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$, prove that

$$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

Long Answer Type

11. If $y = e^x \sin x^3 + (\tan x)^x$, find $\dfrac{dy}{dx}$.

12. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{3}$.

13. Verify Lagrange's mean value theorem for the function $f(x) = x\ (x - 1)\ (x - 2)$ in the interval $\left[0, \dfrac{1}{2}\right]$.

14. (i) If $y = b\tan^{-1}\left(\dfrac{x}{a} + \tan^{-1}\dfrac{y}{x}\right)$, find $\dfrac{dy}{dx}$.

(ii) If $\sqrt{x} + \sqrt{y} = 4$, find $\left.\dfrac{dy}{dx}\right|_{\text{at } x=1}$.

15. (i) If Rolle's theorem hold for the function $f(x) = x^3 + bx^2 + ax + 5$ on $[1, 3]$

where $c = \left(2 + \dfrac{1}{\sqrt{3}}\right)$, find the values of a and b.

(ii) Using Rolle's theorem, find at what points on the curve $y = x^2$ on $[-2, 2]$ is the tangent parallel to x-axis.

SOLUTIONS

1. Since $\sin x$ and $\cos x$ are continuous functions and difference of two continuous functions is a continuous function, therefore $\sin x - \cos x$ *i.e.*, $f(x)$ is a continuous function.

2. Let $y = \cos(\sin x)$

Now, $\dfrac{dy}{dx} = \dfrac{d\{\cos(\sin x)\}}{dx}$

$= -\sin(\sin x) \cdot \cos x = -\cos x \sin(\sin x)$

3. Given, $xy = x^3 + y^3$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3)$$

or $1\cdot y + x\cdot\dfrac{dy}{dx} = 3x^2 + 3y^2\dfrac{dy}{dx}$

$\Rightarrow (x - 3y^2)\dfrac{dy}{dx} = 3x^2 - y \Rightarrow \dfrac{dy}{dx} = \dfrac{3x^2 - y}{x - 3y^2}$

4. $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3}(2x^2 - 1) = 17$

$f(3) = 17$

$\therefore$ f is continuous at $x = 3$.

5. At $x = 1$,

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} x = 1$, $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1} 5 = 5$

$\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x)$

$\therefore$ f is discontinuous at $x = 1$.

6. Here, $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

$\therefore$ L.H.L. $= \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} f(x) = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} \dfrac{k\cos x}{\pi - 2x}$

$= \lim\limits_{h\to 0} \dfrac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim\limits_{h\to 0} \dfrac{k\sin h}{2h}$

$= \lim\limits_{h\to 0} \dfrac{k}{2} \times \dfrac{\sin h}{h} = \dfrac{k}{2} \times 1 = \dfrac{k}{2}$ $\left(\because \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1\right)$

R.H.L. $= \lim\limits_{x\to\left(\frac{\pi}{2}\right)^+} f(x) = \lim\limits_{x\to\left(\frac{\pi}{2}\right)^+} \dfrac{k\cos x}{\pi - 2x}$

$= \lim\limits_{h\to 0} \dfrac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim\limits_{h\to 0} \dfrac{-k\sin h}{-2h}$

$= \lim\limits_{h\to 0} \dfrac{k}{2} \times \dfrac{\sin h}{h} = \dfrac{k}{2} \times 1 = \dfrac{k}{2}$ $\left(\because \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1\right)$

Also, $f\left(\dfrac{\pi}{2}\right) = 3$.

Since, $f(x)$ is continuous at $x = \dfrac{\pi}{2}$

$\therefore$ L.H.L. = R.H.L. $= f\left(\dfrac{\pi}{2}\right) \Rightarrow \dfrac{k}{2} = 3 \Rightarrow k = 6$

7. Here, $y^x = e^{y-x}$

Taking log on both sides, we get

$\log y^x = \log e^{y-x}$

$\Rightarrow$ $x \log y = (y - x) \log e \Rightarrow x \log y = y - x$ ...(i)

On differentiating w.r.t. x, we get

$\left[x\dfrac{d}{dx}(\log y) + \log y \dfrac{d}{dx}(x)\right] = \dfrac{dy}{dx} - 1$

(Using product rule)

$\Rightarrow$ $x\left(\dfrac{1}{y}\right)\dfrac{dy}{dx} + \log y(1) = \dfrac{dy}{dx} - 1$

$\Rightarrow$ $\dfrac{dy}{dx}\left(\dfrac{x}{y} - 1\right) = -1 - \log y$

$\Rightarrow$ $\dfrac{dy}{dx}\left[\dfrac{y}{(1+\log y)y} - 1\right] = -(1 + \log y)$

$\Rightarrow$ $\left[\because \text{ from (i), } x = \dfrac{y}{(1+\log y)}\right]$

$\Rightarrow$ $\dfrac{dy}{dx}\left[\dfrac{1 - 1 - \log y}{1 + \log y}\right] = -(1 + \log y)$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{(1+\log y)^2}{\log y}$

8. We have, $Rf'(1) = \lim\limits_{h\to 0^+} \dfrac{f(1+h) - f(1)}{h}$

$= \lim\limits_{h\to 0^+} \dfrac{[1+h] - [1]}{h} = 0$ ($\because$ $[1 + h] = 1$ and $[1] = 1$)

and $Lf'(1) = \lim\limits_{h\to 0^-} \dfrac{f(1-h) - f(1)}{-h}$

$= \lim\limits_{h\to 0^-} \dfrac{[1-h] - [1]}{-h} = \infty$

$\{\because [1 - h] = 0$ and $[1] = 1\}$.

Thus $Rf'(1) \neq Lf'(1)$.

Hence, $f(x) = [x]$ is not differentiable at $x = 1$.

9. Putting $\sqrt{x} = t, e^{\sqrt{x}} = e^t = u$ ...(i)

we get, $y = \sqrt{e^{\sqrt{x}}} = \sqrt{u}$

$\Rightarrow$ $\dfrac{dy}{du} = \dfrac{1}{2}u^{-1/2} = \dfrac{1}{2\sqrt{u}}$, $\because$ $u = e^t$ $\Rightarrow$ $\dfrac{du}{dt} = e^t$

and $t = \sqrt{x} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$

$\Rightarrow$ $\dfrac{dy}{dx} = \left(\dfrac{dy}{du} \times \dfrac{du}{dt} \times \dfrac{dt}{dx}\right)$

$= \left(\dfrac{1}{2\sqrt{u}} \times e^t \times \dfrac{1}{2\sqrt{x}}\right) = \left\{\dfrac{1}{2\sqrt{u}} \times u \times \dfrac{1}{2\sqrt{x}}\right\}$

$= \dfrac{\sqrt{u}}{4\sqrt{x}} = \dfrac{e^{\frac{1}{2}t}}{4\sqrt{x}} = \dfrac{e^{\frac{1}{2}\sqrt{x}}}{4\sqrt{x}}$. (Using (i))

10. We have, $x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1$ ...(i)

Putting $x = \sin\theta$ and $y = \sin\phi$ in (i), we get

$\sin\theta\cos\phi + \cos\theta\sin\phi = 1$

$\Rightarrow$ $\sin(\theta + \phi) = 1$ $\Rightarrow$ $(\theta + \phi) = \sin^{-1}(1)$

$\Rightarrow \sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$ ...(ii)

On differentiating both sides of (ii) w.r.t. x, we get

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}.$$

11. Let $u = e^x \sin x^3$ and $v = (\tan x)^x$

Now, $u = e^x \sin x^3$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} = e^x \cdot \frac{d\{\sin(x)^3\}}{dx} + \sin x^3 \cdot \frac{d}{dx}(e^x)$$

$$= e^x \cdot \cos x^3 \cdot 3x^2 + \sin x^3 \cdot e^x$$

Hence, $\frac{du}{dx} = 3x^2 \cdot e^x \cos x^3 + e^x \sin x^3$

Again, $v = (\tan x)^x \therefore \log v = x \log (\tan x)$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 1 \cdot \log(\tan x) + x \cdot \frac{1}{\tan x} \sec^2 x$$

$$\therefore \frac{dv}{dx} = v [\log (\tan x) + x \cot x \cdot \sec^2 x]$$

$$= (\tan x)^x [\log (\tan x) + x \cot x \sec^2 x]$$

Now, $y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = 3x^2 e^x \cos (x^3) + e^x \sin (x^3) + (\tan x)^x [\log(\tan x) + x \cot x \sec^2 x]$$

12. $x = 3 \sin t - \sin 3t \Rightarrow \frac{dx}{dt} = 3 \cos t - 3\cos 3t$ ...(i)

$y = 3 \cos t - \cos 3t \Rightarrow \frac{dy}{dt} = -3 \sin t + 3 \sin 3t$ ...(ii)

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin 3t - \sin t}{\cos t - \cos 3t} \text{ [Dividing (ii) by (i)]}$$

$$= \frac{2\cos 2t \sin t}{2 \sin 2t \sin t} = \cot(2t)$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -2\operatorname{cosec}^2 2t \cdot \frac{dt}{dx}$$

$$= -2 \operatorname{cosec}^2 2t \cdot \frac{1}{3(\cos t - \cos 3t)} \text{ [From (i)]}$$

$$\text{At } t = \frac{\pi}{3}, \frac{d^2y}{dx^2} = -2\operatorname{cosec}^2 \frac{2\pi}{3} \cdot \frac{1}{3\left(\cos\frac{\pi}{3} - \cos\frac{3\pi}{3}\right)}$$

$$= -2\left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{1}{3\left(\frac{1}{2}+1\right)} = -\frac{16}{27}$$

13. Given $f(x) = x(x-1)(x-2) = x(x^2 - 3x + 2)$ ...(i)

$\therefore f'(x) = 1.(x^2 - 3x + 2) + x(2x - 3)$

$= 3x^2 - 6x + 2$ ...(ii)

Clearly, $f'(x)$ is finite and unique for all x and hence $f(x)$ is differentiable as well as continuous for all x.

Hence, $f(x)$ is continuous in $\left[0, \frac{1}{2}\right]$.

Also, $f(x)$ is differentiable in $\left(0, \frac{1}{2}\right)$

Hence all conditions of Lagrange's mean value theorem are satisfied for $f(x)$ in $\left[0, \frac{1}{2}\right]$.

From (i), $f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) = \frac{3}{8}$

Now, $f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2}} = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c + 8 = 3 \Rightarrow 12c^2 - 24c + 5 = 0$$

$$\therefore c = \frac{24 \pm \sqrt{576 - 240}}{24} = 1 \pm \frac{\sqrt{21}}{6}$$

Hence, $c = 1 + \frac{\sqrt{21}}{6}, 1 - \frac{\sqrt{21}}{6}$

But $0 < c < \frac{1}{2} \therefore c = 1 - \frac{\sqrt{21}}{6}$

Thus, there exists at least one c $\left(= 1 - \frac{\sqrt{21}}{6}\right)$ in $\left(0, \frac{1}{2}\right)$ such that $f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$

Thus, Lagrange's mean value theorem has been verified.

14. (i) We have, $y = b \tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$

$$\Rightarrow \frac{y}{b} = \tan^{-1}\left(\frac{x}{a} + \tan^{-1}\frac{y}{x}\right)$$

$$\Rightarrow \tan\frac{y}{b} = \frac{x}{a} + \tan^{-1}\frac{y}{x}$$

Differentiating both sides with respect to x, we get

$$\frac{1}{b}\sec^2\frac{y}{b}\frac{dy}{dx}=\frac{1}{a}+\frac{1}{1+(y/x)^2}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{1}{b}\sec^2\frac{y}{b}\frac{dy}{dx}=\frac{1}{a}+\frac{x^2}{x^2+y^2}\frac{x\frac{dy}{dx}-y(1)}{x^2}$$

$$\Rightarrow \quad \frac{1}{b}\sec^2\frac{y}{b}\frac{dy}{dx}=\frac{1}{a}+\frac{x}{x^2+y^2}\frac{dy}{dx}-\frac{y}{x^2+y^2}$$

$$\Rightarrow \quad \left(\frac{1}{b}\sec^2\frac{y}{b}-\frac{x}{x^2+y^2}\right)\frac{dy}{dx}=\frac{1}{a}-\frac{y}{x^2+y^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\frac{1}{a}-\frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\frac{y}{b}-\frac{x}{x^2+y^2}}$$

(ii) We have, $\sqrt{x}+\sqrt{y}=4$ …(i)

Differentiating both sides with respect to x, we get

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0\Rightarrow\frac{1}{2\sqrt{y}}\frac{dy}{dx}=\frac{-1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

Putting $x = 1$ in (i), we get

$$\sqrt{1}+\sqrt{y}=4 \Rightarrow \sqrt{y}=4-1=3\Rightarrow y=9$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{(1,9)}=-\sqrt{\frac{9}{1}}=-3.$$

15. (i) We have

$f(x) = x^3 + bx^2 + ax + 5, x \in [1, 3]$

$\Rightarrow f'(x) = 3x^2 + 2bx + a$

Since Rolle's theorem holds for the function,

$\therefore \quad f'(c) = 0 \Rightarrow 3c^2 + 2bc + a = 0$

$$\Rightarrow \quad c=\frac{-2b\pm\sqrt{4b^2-12a}}{6}=\frac{-b\pm\sqrt{b^2-3a}}{3}$$

$$\Rightarrow \quad 2+\frac{1}{\sqrt{3}}=\frac{-b\pm\sqrt{b^2-3a}}{3}$$

$$\Rightarrow \quad 2+\frac{1}{\sqrt{3}}=-\frac{b}{3}+\frac{\sqrt{b^2-3a}}{3}$$

$$\Rightarrow \quad 2=-\frac{b}{3} \text{ and } \frac{\sqrt{b^2-3a}}{3}=\frac{1}{\sqrt{3}}$$

$\Rightarrow b = -6$ and $b^2 - 3a = 3$

$\Rightarrow b = -6$ and $a = 11$

(ii) Let $f(x) = x^2$

(a) $f(x) = x^2$, being a polynomial, is a continuous function on $[-2, 2]$.

(b) $f'(x) = 2x$ which exists in $(-2, 2)$

$\therefore$ $f(x)$ is derivable in $(-2\ 2)$.

(c) Also, $f(-2) = f(2) = 4$

Thus, all the conditions of Rolle's theorem are satisfied. Hence there must exist at least one value $c \in (-2, 2)$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \Rightarrow 2c = 0$ $[\because f'(x) = 2x]$

$\Rightarrow \quad c = 0 \in (-2, 2)$

Thus, the tangent to the curve is parallel to x-axis at $x = 0$.

At $x = 0, y = 0$. $[\because \ y = x^2]$

$\therefore$ Tangent to the curve $y = x^2$ is parallel to x-axis at $(0,0)$.

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1. If $\vec{x}\times\vec{y}=\vec{a}, \vec{y}\times\vec{z}=\vec{b}, \vec{x}\cdot\vec{b}=\gamma, \vec{x}\cdot\vec{y}=1$ and $\vec{y}\cdot\vec{z}=1$, then find $\vec{x},\vec{y},\vec{z}$ in terms of $\vec{a},\vec{b}$ and γ.

[Yukti Khurana, Punjab]

Ans. Given, $\vec{x}\times\vec{y}=\vec{a}$ …(1)

$\vec{y}\times\vec{z}=\vec{b}$ …(2)

and $\vec{x}\cdot\vec{b}=\gamma,\ \vec{x}\cdot\vec{y}=1,\ \vec{y}\cdot\vec{z}=1$ …(3)

From (1), $(\vec{x}\times\vec{y})\times\vec{b}=\vec{a}\times\vec{b}$

$\Rightarrow (\vec{x}\cdot\vec{b})\,\vec{y}-(\vec{y}\cdot\vec{b})\,\vec{x}=\vec{a}\times\vec{b}$

$\Rightarrow \gamma\vec{y}-(\vec{y}\cdot\vec{b})\vec{x}=\vec{a}\times\vec{b}$ $[\because\ \vec{x}\cdot\vec{b}=\gamma]$ …(4)

From (2), $\vec{y}\cdot\vec{b}=\vec{y}\cdot(\vec{y}\times\vec{z})=0$

$\therefore$ From (4), $\vec{y}=\dfrac{\vec{a}\times\vec{b}}{\gamma}$ …(5)

From (1), taking cross product with $\vec{y}$, we get

$(\vec{x}\times\vec{y})\times\vec{y}=\vec{a}\times\vec{y}$

$\Rightarrow (\vec{x}\cdot\vec{y})\,\vec{y}-(\vec{y}\cdot\vec{y})\,\vec{x}=\vec{a}\times\vec{y}$

$\Rightarrow \vec{y}-|\vec{y}|^2\,\vec{x}=\vec{a}\times\vec{y}$ $[\because\ \vec{x}\cdot\vec{y}=1]$

$\Rightarrow \vec{x}=\dfrac{1}{y^2}[\vec{y}-\vec{a}\times\vec{y}]$, where $\vec{y}=\dfrac{\vec{a}\times\vec{b}}{\gamma}$

From (2), taking cross product with $\vec{y}$, we get

$(\vec{y}\times\vec{z})\times\vec{y}=\vec{b}\times\vec{y}$

$\Rightarrow y^2\vec{z}-(\vec{z}\cdot\vec{y})\vec{y}=\vec{b}\times\vec{y}$

$\Rightarrow y^2\vec{z}-\vec{y}=\vec{b}\times\vec{y}$ $[\because\ \vec{y}\cdot\vec{z}=1]$

$\Rightarrow \vec{z}=\dfrac{1}{y^2}[\vec{y}+\vec{b}\times\vec{y}]$, where $\vec{y}=\dfrac{\vec{a}\times\vec{b}}{\gamma}$

2. If $\tan^{-1}y : \tan^{-1}x = 4 : 1$, express y as an algebraic function of x. Hence or otherwise prove that $\tan 22\dfrac{1^\circ}{2}$ is a root of the equation $x^4 + 1 = 6x^2$.

[Ramesh Kr, Patna]

Ans. Here, $\tan^{-1}y = 4\tan^{-1}x = 2(2\tan^{-1}x)$

$$= 2\tan^{-1}\frac{2x}{1-x^2}$$

$$= \tan^{-1}\frac{2\cdot\frac{2x}{1-x^2}}{1-\left(\frac{2x}{1-x^2}\right)^2} = \tan^{-1}\frac{4x(1-x^2)}{(1-x^2)^2-4x^2}$$

or $y=\dfrac{4x(1-x^2)}{1+x^4-6x^2}$

Let $\tan^{-1}x = 22\dfrac{1^\circ}{2}$; then $4\tan^{-1}x = 90^\circ$

$\Rightarrow \tan^{-1}y = 90^\circ$ $\{\because\ \tan^{-1}y = 4\tan^{-1}x\}$

$\Rightarrow y = \tan 90^\circ = \dfrac{4x(1-x^2)}{1+x^4-6x^2}$

$\Rightarrow 1 + x^4 - 6x^2 = 0$

$\Rightarrow 1 + x^4 = 6x^2$

$\therefore$ $x = \tan 22\dfrac{1^\circ}{2}$ is a root of the equation $1 + x^4 = 6x^2$.

3. Prove that

$(b - a)\sec^2 a < \tan b - \tan a < (b - a)\sec^2 b$,

where $0 < a < b < \dfrac{\pi}{2}$.

[Chandan Biswas, Warangal]

Ans. Consider the function $f(x) = \tan x$, $x \in (a, b)$.

Clearly, $f(x)$ is continuous and differentiable in $\left(0, \dfrac{\pi}{2}\right)$ and so in (a, b).

Therefore, by Lagrange's theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c), \text{ where } a < c < b$$

$$\Rightarrow \frac{\tan b-\tan a}{b-a} = \sec^2 c$$

or $\tan b - \tan a = (b - a)\sec^2 c$ …(i)

But $a < c < b \Rightarrow \sec a < \sec c < \sec b$

$\therefore$ $(b - a)\sec^2 a < (b - a)\sec^2 c < (b - a)\sec^2 b$

$\Rightarrow$ $(b - a)\sec^2 a < \tan b - \tan a < (b - a)\sec^2 b$. (from (i))

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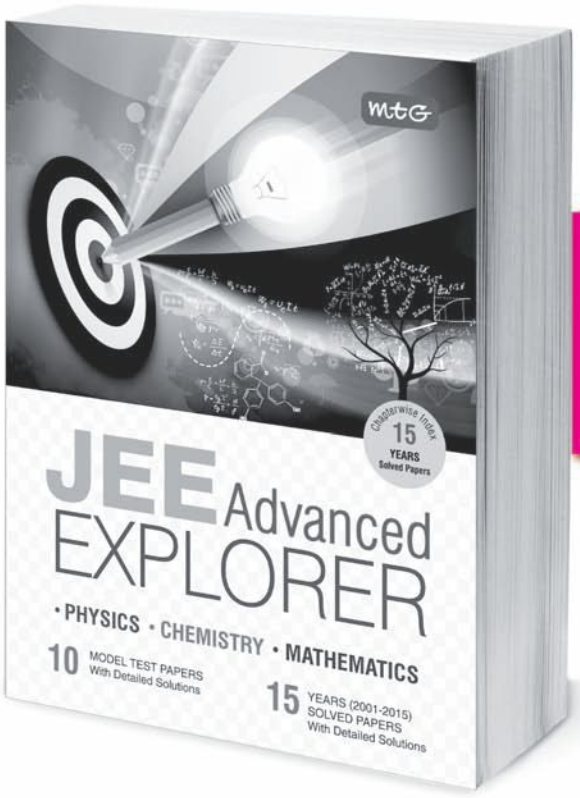

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1. When 42 is added to one number, it increases by 30 percent. What is 40 percent of that number?
(a) 48 (b) 64
(c) 42 (d) 56

2. On a tour, a man travels at the rate of 64 km an hour for the first 160 km. He then travels the next 160 km at the rate of 80 km an hour. What is the average speed (in km per hour) for the first 320 km of the tour?
(a) 71.11 (b) 72
(c) 35.55 (d) 36

3. The length of a rectangle is increased by 60%. By what percent would the width have to be decreased to maintain the same area?
(a) 60 (b) 75
(c) 120 (d) $37\frac{1}{2}$

4. A train 50 metres long passes a platform in 10 seconds. What is the speed of the train in metres /second?
(a) 50 (b) 10
(c) 15 (d) data inadequate.

5. A sum of money at simple interest amounts to Rs. 2800 in 2 years and to Rs. 3250 in 5 years at the rate of
(a) 8% (b) 6%
(c) 5% (d) 7%.

6. In a box, there were a dozen cakes of soap in each horizontal and vertical row. If there were 12 such layers in the box, find the total number of soap cakes in the box.
(a) 36 (b) 144
(c) 1728 (d) 432.

7. A fruit-seller buys apples at the rate of Rs. 10 per dozen and sells 10 for Rs. 12. What percent of profit does he make?
(a) 44 % (b) 33 %
(c) 30 % (d) 27 %

8. When the sum of two numbers is multiplied by 5, the product is divisible by 15. Which one of the following could be the pair of the numbers?
(a) 240,335 (b) 250,341
(c) 272,347 (d) 245,342.

9. Rs. 8840 is divided between x and y in such a manner that for every Rs. 6 that x gets, y gets Rs. 7. Their shares are
(a) Rs. 4080, Rs. 4760 (b) Rs. 5328, Rs. 2411
(c) Rs. 4180, Rs. 4660 (d) Rs. 4280, Rs. 4560.

10. A's shop's sale of chairs increases every month by 10 percent. If the sale for May was 1,000, what would be its approximate expected sale in August?
(a) 1210 (b) 1300
(c) 1330 (d) 1331

11. A machine part is packed in a box of size 5 cm × 4.5 cm × 3 cm. The number of boxes that can be packed in a carton of size of 55 cm × 45 cm × 24 cm is
(a) 990 (b) 880
(c) 770 (d) 660

12. 8 children and 12 men complete a certain piece of work in 9 days. Each child takes twice the time taken by a man to finish the work. In how many days will 12 men finish the same work?
(a) 8 (b) 9
(c) 12 (d) 15

13. The ratio of the age of mother to that of son is 7 : 3. If the sum of their ages is 60 years, what is the difference in their ages in years?
(a) 4 (b) 18
(c) 24 (d) 36

14. What will be the amount payable after 2 years if Rs. 10,000 is borrowed at the rate of 10% p.a. compound interest?
(a) Rs. 11,000 (b) Rs. 11,100
(c) Rs. 11,200 (d) Rs. 12,100

15. A cricketer has completed 10 innings and his average is 22.5 runs. How many runs must he make in the next innings so as to raise his average to 25 runs?
(a) 60 (b) 56
(c) 48 (d) 50

16. In a college $(1/5)^{th}$ of the girls and $(1/8)^{th}$ of the boys took part in an annual celebration. What part of the total number of students in the college took part in the celebration?
(a) 13/80 (b) 2/13
(c) 13/40 (d) data inadequate

17. The ratio between the present age of Anup and his age four years hence is 4 : 5. The ratio between the present ages of Anup and Tamana is 4 : 7. If the difference between present age of Anup and Tamana is 18 years, what was Anup's age five years back?
(a) 22 yrs. (b) 38 yrs.
(c) 19 yrs. (d) 20 yrs.

18. An amount of money is to be distributed among x, y and z in the ratio 3 : 7 : 11. The difference between y's and z's shares is Rs. 1600. What is the total of x's and y's share?
(a) Rs. 5,600 (b) Rs. 4,800
(c) Rs. 4,400 (d) Rs. 4,000.

19. Two numbers are such that the ratio between them is 3 : 5 but if each is increased by 10, the ratio between them becomes 5 : 7. Find the numbers.
(a) 15, 25 (b) 7, 9
(c) 13, 22 (d) 3, 5.

20. A reduction of 20% in the price of pen enables a person to purchase 12 more for Rs. 15. What was the price of 16 pens before reduction?
(a) 5 (b) 6
(c) 7 (d) 9.

21. If the difference between selling a shirt at a profit of 10% and 15% is Rs. 10, then its cost price is Rs.
(a) 115 (b) 150
(c) 110 (d) 200

22. A cricketer scored an average of 44 runs in 10 innings. He played one more innings and the average came down to 40 runs. His score in the 11^{th} innings is
(a) 3 (b) 2
(c) 1 (d) 0

23. Saroj bought 18 dozen of apples at the rate of Rs. 198 per dozen. He sold these apples at the rate of Rs. 18 per piece. Approximately, what was his percentage profit?
(a) 5 % (b) 9 %
(c) 6 % (d) 12 %

24. The area of the four walls of a room is 660 m^2 and its length is twice its breadth. If the height of the room is 11 m, then area of its floor (in m^2) is
(a) 120 (b) 150
(c) 200 (d) 330.

25. Ram and Shyam started business investing Rs. 15000 and Rs. 22500 respectively. In which proportion will the profit be distributed?
(a) 1 : 4 (b) 1 : 3
(c) 1 : 2 (d) 2 : 3.

26. Ratio of the present age of Rahim and his daughter is 3 : 1. After 5 years the respective ratio would be 7 : 3. What is the present age of Rahim's daughter?
(a) 5 yrs. (b) 10 yrs.
(c) 15 yrs. (d) 11 yrs.

27. A mat of 26 metres × 19 metres was bought at the cost of Rs. 68666. What is its cost per square metre?
(a) 139 (b) 145
(c) 1525 (d) 2641.

28. Ranjeet spends 15% of his salary on clothes, 30% on food and 10% on transport. After this, if he is left with Rs. 900, then what is his salary?
(a) Rs. 1500 (b) Rs. 1635
(c) Rs. 1800 (d) none of these.

29. A sum lent at 7% per annum will produce as much simple interest in 2 years as Rs. 1750 lent at 5% p.a. will produce in 4 years. The sum is Rs.
(a) 2000 (b) 2200
(c) 2400 (d) 2500.

30. The ratio of the number of ladies to that of gents at a party was 3 : 2. When 20 more gents joined the party, the ratio got reversed. The number of ladies present at the party was
(a) 16 (b) 24
(c) 32 (d) 36.

31. A dealer offered a machine for sale for Rs. 27500 but even if he had charged 10% less, he would have made a profit of 10%. The actual cost of the machine is
(a) Rs. 22,000 (b) Rs. 24,250
(c) Rs. 22,500 (d) Rs. 22,275.

32. The ratio of girls to boys in a school is 4 : 5. If the number of boys is 200 more than that of girls, what is the number of boys in the school?
(a) 200 (b) 1200
(c) 800 (d) 1000.

33. A car takes 3 hours to travel from New Delhi to Jaipur. How long will a train take to travel one-fourth of the distance at an average speed which is one and a half times that of the car?
(a) 15 min. (b) 30 min.
(c) 25 min. (d) 45 min.

34. There are 20 row of chairs in a hall. Each row has an equal number of chairs. If there are 30 chairs in 2 rows, then the total number of chairs in the hall are
(a) 600 (b) 1200
(c) 300 (d) 200.

35. The population of a village is 4,500. $(5/9)^{th}$ of them are males and rest are females. If 40% of the males are married, then the percentage of married females is
(a) 35 % (b) 40 %
(c) 50 % (d) 60 %

36. After John gave Rs. 12 to Hari and Hari gave Rs. 4 to John, both had equal amount with them. Who had the smaller amount initially and by how much?
(a) Hari by Rs. 12 (b) John by Rs. 4
(c) Hari by Rs. 8 (d) none of these.

37. Anup is three years younger than Kumar. Anil is two years older than Anup. What is Kumar's age in relation to Anil?
(a) 3 years old (b) 4 years younger
(c) 4 years older (d) 1 year older.

38. A retailer buys a radio for Rs. 215. His overhead expenses are Rs. 25. If he sells the radio for Rs. 300, his profit percentage will be
(a) 20% (b) 25%
(c) 39.5% (d) 43.6%.

39. If three-seventh of one-third of four-ninth of a number is 12, what is the number?
(a) 158 (b) 179
(c) 126 (d) 189.

40. The simple interest on a sum of money at the end of eight years amounts to its 40 per cent. What is the rate of interest per cent per annum?
(a) 6 % (b) 5 %
(c) 10 % (d) 7 %.

41. Study the table given below and find in which year the number of students qualified in Hindi was two hundred percent of the number of students qualified in English in the same year.

Years	Subjects			
	English		Hindi	
	Applied	Qualified	Applied	Qualified
1990	196	84	140	30
1991	212	34	150	40
1992	160	40	210	60
1993	190	35	240	70
1994	200	47	260	80

(a) 1990 (b) 1991 (c) 1992 (d) 1993.

42. When the price of a TV set was increased by 30%, the number of TV sets sold decreased by 20%. What was the effect on revenue?
(a) 8% decrease (b) 8% increase
(c) 4% increase (d) 4% decrease.

43. Ramesh purchased a plot of land at Rs. 80 per sq. metre through an estate dealer. If the area of the plot was 300 sq. metres and the rate of commission was 3%, what amount was Ramesh required to pay in all?
(a) Rs. 24,900 (b) Rs. 24,720
(c) Rs. 23,100 (d) Rs. 23,280.

44. The average salary of 20 workers in an office is Rs. 1900 per month. If the manager's salary is added, the average salary becomes Rs. 2000 per month. What is the Manager's annual salary?
(a) Rs. 45,600 (b) Rs. 24,000
(c) Rs. 25,200 (d) none of these.

45. A person purchased an article and sold it at a loss of 10%. If he had bought it for Rs. 20% less and sold it for Rs. 55 more, he would have made a profit of 40%. The cost price of the article is
(a) Rs. 225 (b) Rs. 250
(c) Rs. 275 (d) Rs. 300.

ANSWER KEYS

1. (d)	**2.** (a)	**3.** (d)	**4.** (d)	**5.** (b)	**6.** (c)
7. (a)	**8.** (b)	**9.** (a)	**10.** (d)	**11.** (b)	**12.** (c)
13. (c)	**14.** (d)	**15.** (d)	**16.** (d)	**17.** (c)	**18.** (d)
19. (a)	**20.** (a)	**21.** (d)	**22.** (d)	**23.** (b)	**24.** (c)
25. (d)	**26.** (b)	**27.** (a)	**28.** (d)	**29.** (d)	**30.** (b)
31. (c)	**32.** (d)	**33.** (b)	**34.** (c)	**35.** (c)	**36.** (d)
37. (d)	**38.** (b)	**39.** (d)	**40.** (b)	**41.** (d)	**42.** (c)
43. (b)	**44.** (d)	**45.** (b)			

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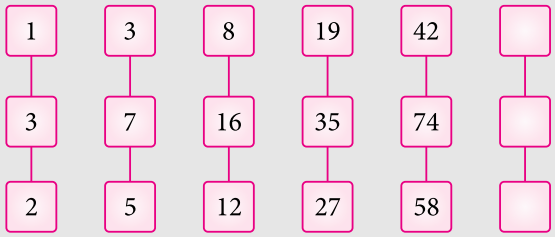
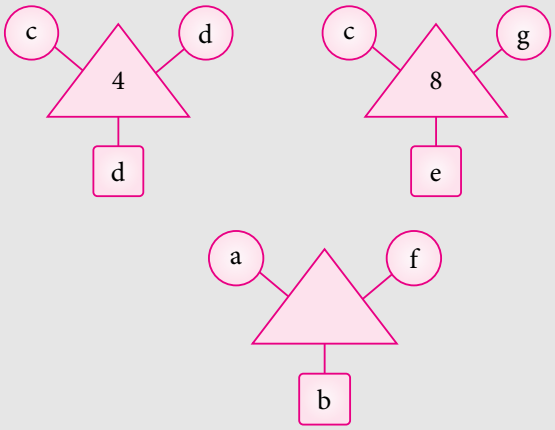
MIND BENDERS

1 What are the missing numbers?



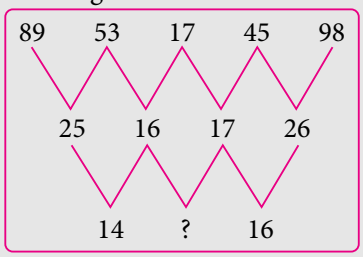
Ans. 89, 153, 121. Every number is multiplied by 2 and then added to : +1, +2, +3, +4, +5.

2 What number is missing from the triangle?



Ans. 2. Multiply the letters in the circles, subtract the number in the square, and divide the sum by 2.

3 In the given puzzle, the numbers in the second row are determined by the relationships of the numbers in the first row. Likewise, the numbers in the third row are determined by the relationships of the numbers in the second row. Can you determine the relationships and find the missing number?



Ans. 15. Simply add the sum of the two digits in any number to the sum of the two digits in the adjacent number to get the corresponding number in the row below. For example :

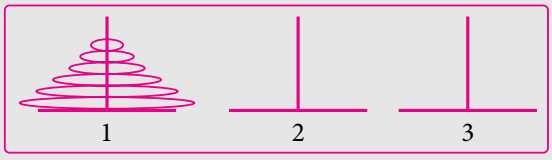
8 + 9 (89) and 5 + 3 (53) = 25

5 + 3 (53) and 1 + 7 (17) = 16

To find the missing number, add:

1 + 6 (16) and 1 + 7 (17) = 15

4 Below are six discs stacked on a peg. The object is to reassemble the discs, one by one, in the same order on another peg, using the smallest number of moves. No larger disc can be placed on a smaller disc. How many moves will it take?



Ans. It will take 63 moves. For any number of discs n, the number of moves can be found by using the formula $2^n - 1$.

5 A shepherd says to another, "If I give you one sheep, you will have twice the number of sheep that I have, but if you give me one, we will both

have the same number of sheep." How many sheep did each shepherd have?

Ans. The shepherd that is talking had 5 sheep and the other one had 7.

6 Yesterday afternoon, I went to visit my friend Albert, who is a painter. While I was watching him paint, I told him, "No wonder it takes you so long to finish a painting. Since I arrived, you have entered the studio twelve times." How many times did he leave the studio?

Ans. 11 times (one fewer than the number of times he went in).

7 A spider spins its web in a window frame. Each day, it spins an area equal to that of the amount already completed. It takes 30 days to cover the entire window frame. How long would two spiders take? (In the case of two spiders, each of them spins an amount equal to the area of the existing part of the web made by that particular spider.)

Ans. 29 days. One spider would have covered half of the space on the 29th day, and on the 30th day would repeat what had been done, covering the space completely. Two spiders would each have covered half of the space in 29 days, therefore covering the entire area.

8 Suppose all counting numbers were arranged in columns as shown below. Under what letter would the number 100 appear?

A	B	C	D	E	F	G
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	—	—	—	—

Ans. It would appear in column B. Divide by 7 whatever number you wish to place, and see what the remainder is. If the remainder is 1, the number goes in column A; if the remainder is 2, the number goes in column B; and so on. (If the remainder is zero, however, the number goes in column G.)

9 We have three boxes. One contains two black marbles, the second box contains two white marbles, and the third box contains one black and one white marble. The boxes are marked BB, WW, BW. However, no code corresponds to the marbles in its box. What would be the smallest number of marbles that must be randomly picked, from one or several boxes, to identify their contents?

Ans. Take one marble from the box marked BW. If it is white, the other marble must also be white. This means that the box marked BB must have black and white marbles and the box marked WW must have only black marbles. You can apply the same principle if the first marble you take is black.

10 Sarah is older than Julie and Maggie. Maggie is older than Paula. Ann is younger than Julie, but older than Paula. Ann is younger than Maggie. Sarah is younger than Liz. Who is the second oldest woman in this group?

Ans. Sarah is the second oldest; Liz is the oldest.

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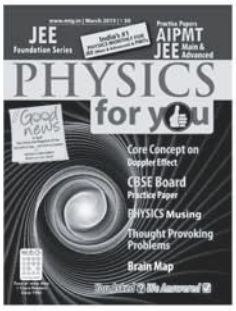

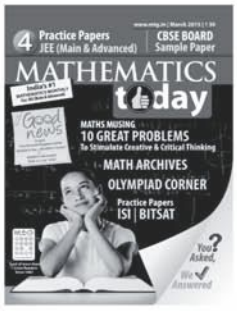

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